Idempotents of $M_2(Z_{10}[x])$

Priyanka, Sanyogita

Department of Mathematics, C.R.M Jat College, Hisar

ABSTRACT

The aim of this paper is to study idempotents in the matrix ring $M_2(Z_{10}[x])$

1 INTRODUCTION

Idempotents in rings play a very important role in the study of rings. The term “idempotents” was introduced by Benjamin Peirce in the context of elements of Algebra that remain invariant when raised to positive integer power and the literally means, “the same power”, from idem+potence (same+power).

In Boolean algebra, the main subject of study are rings in which elements are idempotents under both addition and multiplication. In the case of polynomial rings, Kanwar, Leroy and Matczuk showed that for an abelian ring $R$, idempotents in the polynomial ring $R[x]$ over $R$ are precisely idempotents in $R$ ([7], Lemma 1). In fact, a ring is reduced if and only if the unit group of $R[x]$ is same as the unit group of $R$.

In this article, we study idempotents in matrix ring $M_2(Z_{10}[x])$. Throughout, a ring will mean an associative ring with unity and for any positive integer $n$, $Z_n$ will denote the ring of integers modulo $n$. For any ring $R$, $E(R)$ will denote the set of all idempotents in $R$. For any positive integer $n$, $M_n(R)$ will denote the ring of $n \times n$ matrices over a ring $R$ and $GL(n,R)$ will denote the general linear group (the group of all $n \times n$ invertible matrices over ring $R$).

We will use standard definitions for determinant and trace of matrices over commutative ring. More precisely for a $2 \times 2$ matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ over a commutative ring $R$, determinant of $A$ is $ad - bc$ and trace of $A$ is $a + d$. Recall that the determinant of product of two matrices over a commutative ring is the product of the determinant of two matrices.

2 IDEMPOTENTS OF $M_2(Z_{10}[x])$

We now give some results that will be useful in our study. We begin with the following proposition that may also be of independent interest.

Proposition 2.1. Let $R$ be any ring with unity and $a = \sum a_i x^i$ is an element in $R[x]$ such that $a^2 - a \in R$. If any of the following conditions hold:

1. $R$ has no non-zero nilpotent elements,
2. $a \neq 0$ for $1 \leq i \leq n$ and $2a_0 - 1$ is a unit in $R$, then $a \in R$.

Proof.

If $R$ has no non-zero nilpotent elements and $a^2 - a \in R$, then it is easy to see that $a_i = 0$ for $1 \leq i \leq n$. The proof, in the second case, is similar to the proof of Lemma 1 in [7]. We give a brief outline for the sake of completeness. If $a \neq 0$ and $a_i$ is the first non-zero coefficient in $a$, then $a^2 - a \in R$ gives $2a_0 a_i - a_i = 0$. But then $a_i = 0$ as $2a_0 - 1$ is a unit in $R$, a contradiction. Thus $a \in R$.

In particular, we have the following corollary.

Corollary 2.2. [7, Lemma 1] If $R$ is a commutative ring, then $E(R[x]) = E(R)$.

Corollary 2.3. If $R$ is a ring with no non-zero nilpotent elements, then $E(R[x]) = E(R)$.

Theorem 2.4. Any non-trivial idempotent in $M_2(Z_{10}[x])$ is of one of the following forms:

1. $\begin{bmatrix} a(x) & b(x) \\ c(x) & 1 - a(x) \end{bmatrix}$, where $a(x)(1-a(x)) = b(x)c(x)$
2. $\begin{bmatrix} 5a(x) & 5b(x) \\ 5c(x) & 5 - 5a(x) \end{bmatrix}$, where $a(x)(5-5a(x)) - 5b(x)c(x) = 2f(x)$
3. \[
\begin{bmatrix}
2a(x) & 2b(x) \\
2c(x) & 6 - 2a(x)
\end{bmatrix},
\text{where } 2a(x)(6 - 2a(x)) - 4b(x)c(x) = 0
\]
4. \[
\begin{bmatrix}
2a(x) + 1 & 2b(x) \\
2c(x) & 5 - 2a(x)
\end{bmatrix},
\text{where } (2a(x) + 1)(5 - 2a(x)) - 4b(x)c(x) = 5
\]
5. \[
\begin{bmatrix}
1 & 5b(x) \\
5c(x) & 6
\end{bmatrix},
\text{where } 6 - 5b(x)c(x) = 6
\]
6. \[
\begin{bmatrix}
6 & 5b(x) \\
5c(x) & 1
\end{bmatrix},
\text{where } 6 - 5b(x)c(x) = 6
\]
7. \[
\begin{bmatrix}
2 & 5b(x) \\
5c(x) & 7
\end{bmatrix},
\text{where } 4 - 5b(x)c(x) = 6
\]
8. \[
\begin{bmatrix}
7 & 5b(x) \\
5c(x) & 2
\end{bmatrix},
\text{where } 4 - 5b(x)c(x) = 6
\]
9. \[
\begin{bmatrix}
5 & 0 \\
0 & 6
\end{bmatrix},
\text{where } a(x), b(x), c(x), f(x) \text{ are polynomial in } \mathbb{Z}_{10}[x], \text{not necessarily non-zero.}
\]

proof:
Since the idempotents in \( \mathbb{Z}_{10}[x] \) are precisely the idempotents in \( \mathbb{Z}_{10} \). Therefore, the idempotents in \( \mathbb{Z}_{10}[x] \) are \( 0, 1, 5, \) and \( 6 \). Now let \( A = \begin{bmatrix} a(x) & b(x) \\ c(x) & d(x) \end{bmatrix} \) be a non-trivial idempotent of \( M_2(\mathbb{Z}_{10}[x]) \). For convenience, we will write \( a, b, c, d \) for \( a(x), b(x), c(x), d(x) \) respectively. Since \( A \) is an idempotent, we have \( a^2 + bc = a, b(a + d) = b, c(a + d) = c, \) and \( bc + d^2 = d \). Also since determinant of \( A \) is an idempotent in \( \mathbb{Z}_{10} \), so the determinant of \( A \) is \( 0 \) or \( 1 \) or \( 5 \) or \( 6 \).

If determinant of \( A \) is \( 1 \) then \( A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \), a trivial idempotent in \( M_2(\mathbb{Z}_{10}[x]) \). Hence, the determinant of \( A \) is \( 0 \) or \( 5 \) or \( 6 \). Also, trace of \( A \) is in \( \mathbb{Z}_{10} \), that is, \( a + d \) is in \( \mathbb{Z}_{10} \).

Case 1: Determinant of \( A \) is \( 0 \). This means \( ad - bc = 0 \). Since \( A \) is an idempotent, therefore, \( a^2 + bc + bc + d^2 = a^2 + 2bc + d^2 = a + d \). Thus, \( a + d \) is an idempotent in \( \mathbb{Z}_{10}[x] \). Thus \( a + d \) is either \( 0 \) or \( 1 \) or \( 5 \) or \( 6 \).

If \( a + d = 0 \), then we get \( A \) to be a zero matrix, which is a trivial idempotent in \( M_2(\mathbb{Z}_{10}[x]) \).

If \( a + d = 1 \) then \( d = 1 - a \) and hence \( ad - bc = 0 \) gives \( a^2 + bc = a \). Also \((a+d)b = b\) and \((a+d)c = c\), and \( bc + d^2 = 1 - a \). Thus, \( A^2 = \begin{bmatrix} a & b \\ c & 1 - a \end{bmatrix} \). Thus in this case, matrix \( A = \begin{bmatrix} a(x) & b(x) \\ c(x) & 1 - a(x) \end{bmatrix} \), where \( a(x), b(x), c(x) \) are polynomial in \( \mathbb{Z}_{10}[x] \), such that \( a(x)(5 - 5a(x)) - \) \( b(x)c(x) = 2f(x) \) for some polynomial \( f(x) \in \mathbb{Z}_{10}[x] \).

Hence, \( A = \begin{bmatrix} 5a(x) & 5b(x) \\ 5c(x) & 5 - 5a(x) \end{bmatrix} \), where \( a(x), b(x), c(x) \in \mathbb{Z}_{10}[x] \) such that \( a(x)(5 - 5a(x)) - b(x)c(x) = 2f(x) \) for some \( f(x) \in \mathbb{Z}_{10}[x] \). If \( a + d = 5 \) then \( d = 5 - a \) and hence \( ad - bc = 0 \) gives \( a^2 + bc = 5a \) and so, \( 4a = 0 \). Also, \((a+d)b = b\) implies \( 4b = 0 \) and \((a+d)c = c\) implies \( 4c = 0 \). Therefore, \( a = 5a_0(x), b = 5b_0(x), \) and \( c = 5c_0(x) \), where \( a_0(x), b_0(x), c_0(x) \) are polynomial in \( \mathbb{Z}_{10}[x] \). Now, since \( ad - bc = 0 \), we get \( 5a_0(x)(5 - 5a_0(x)) = 5b_0(x)c_0(x) \), which is equivalent \( a_0(x)(5 - 5a_0(x)) = b_0(x)c_0(x) = 2f(x) \) for some polynomial \( f(x) \in \mathbb{Z}_{10}[x] \).

If \( a + d = 6 \) then \( d = 6 - a \) and hence \( ad - bc = 0 \) gives \( a^2 + bc = 6a \). Thus, \( 5a = 0 \). Also, \((a+d)b = b\) gives \( 5b = 0 \) and \((a+d)c = c\) gives \( 5c = 0 \). Therefore, \( a = 2a_0(x), b = 2b_0(x), \) and \( c = 2c_0(x) \) and \( d = 6 - 2a_0(x) \), where \( a_0(x), b_0(x), c_0(x) \) are polynomial in \( \mathbb{Z}_{10}[x] \). Now, since \( ad - bc = 0 \), we get \( 2a_0(x)(6 - 2a_0(x)) = 4b_0(x)c_0(x) \). Hence, idempotent matrix is \( A = \begin{bmatrix} 2a(x) & 2b(x) \\ 2c(x) & 6 - 2a(x) \end{bmatrix} \), where \( 2a(x)(6 - 2a(x)) = 4b(x)c(x) \).
Next, we consider the case when determinant of $A$ is 5.

**Case 2:** Determinant of $A$ is 5. This means $ad - bc = 5$. That is, $2ad - 2bc = 0$. Since $A$ is an idempotent, therefore, $a^2 + bc + b^2 + d^2 = a^2 + 2bc + d^2 = a + d$. It means $a + d$ is an idempotent in $\mathbb{Z}_{10}[x]$. Thus, $a + d$ is either 0 or 1 or 5 or 6.

If $a + d = 0$, then $ad - bc = 5$ gives $a^2 + bc = 5$ and so, $a = 5$. Also, $b(a + d) = b$ gives $b = 0$ and $c(a + d) = c$ gives $c = 0$. Thus, $a + d = 0$ gives $d = 5$. Hence, idempotent matrix $A = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$.

If $a + d = 1$, then $d = 1 - a$, so, $ad - bc = 5$ gives $a^2 + bc = a - 5$. Also, $a^2 + bc = a$, so we have $a = a - 5$, which cannot be possible in $\mathbb{Z}_{10}[x]$. If determinant of idempotent matrix $A$ in $\mathbb{Z}_{10}[x]$ is 5, then its trace can’t be 1.

If $a + d = 5$, then $d = 5 - a$, so $ad - bc = 5$ implies $4a = 5$, which is again not possible in $\mathbb{Z}_{10}[x]$. So, this case also fails.

If $a + d = 6$, then $ad - bc = 6$ implies $5a = 5$, so $a$ can be 1, 3, 5, 7, 9 in $\mathbb{Z}_{10}[x]$. Also, $b(a + d) = b$ gives $5b = 0$ and $c(a + d) = c$ gives $5c = 0$. Therefore, $a = 2a(x) + 1$, $b = 2b(x)$, $c = 2c(x)$, and $d = 6 - (2a(x) + 1)$, where $a(x), b(x), c(x)$ are polynomials in $\mathbb{Z}_{10}[x]$. Since $ad - bc = 5$, so

$$A = \begin{bmatrix} 2a(x) + 1 & 2b(x) \\ 2c(x) & 5 - 2a(x) \end{bmatrix}$$

where $a(x), b(x), c(x)$ are polynomials in $\mathbb{Z}_{10}[x]$ such that $2(a(x) + 1) - 4b(x)c(x) = 5$.

**Case 3:** when determinant of $A$ is 6. This means $ad - bc = 6$, so we get $a^2 + bc + b^2 + d^2 = a^2 + 6 + d^2 = (a + d)^2 - 2$. From this, we conclude that trace of matrix $A$ is an idempotent iff $a + d = 2$ or 4 or 7 or 9.

If $a + d = 2$, then $ad - bc = 6$ gives $a = 6$. And $(a + d)b = b$ gives $b = 0$ and $(a + d)c = c$ gives $c = 0$. Also, $a = 6$ gives $d = 5$. Hence, matrix $A = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$.

If $a + d = 4$, then $ad - bc = 6$ gives $a = 2$. And $(a + d)b = b$ gives $3b = 0$, which implies $b = 0$ in $\mathbb{Z}_{10}[x]$. Also, $c(a + d) = c$ gives $c = 0$ in $\mathbb{Z}_{10}[x]$. Hence, matrix $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, whose determinant is 4 while, in this case $ad - bc = 6$. So, this case is not possible.

If $a + d = 7$, then $ad - bc = 6$ gives $6a = 6$, i.e., $a = 1$ or 6 in $\mathbb{Z}_{10}[x]$. Also, $(a + d)b = b$ and $(a + d)c = c$ gives $6b = 0$ and $6c = 0$ respectively. Therefore, $b = 5b(x)$ and $c = 5c(x)$, where $b(x)$ and $c(x)$ are polynomial in $\mathbb{Z}_{10}[x]$. Since $ad - bc = 6$, so, $6 - 5b(x)c(x) = 6$. Hence, matrix

$$A = \begin{bmatrix} 1 & 5b(x) \\ 5c(x) & 6 \end{bmatrix}$$

or $A = \begin{bmatrix} 6 & 5b(x) \\ 5c(x) & 1 \end{bmatrix}$, where $6 - 5b(x)c(x) = 6$, where $b(x), c(x) \in \mathbb{Z}_{10}[x]$.

If $a + d = 9$, then $ad - bc = 6$ implies $8a = 6$, i.e., $a = 2$ or 7 in $\mathbb{Z}_{10}[x]$. Also, $(a + d)b = b$ and $(a + d)c = c$ gives $8b = 0$ and $8c = 0$ respectively. Hence, $b = 5b(x)$ and $c = 5c(x)$, where $b(x)$ and $c(x)$ are polynomial in $\mathbb{Z}_{10}[x]$. Since $ad - bc = 6$, so

$$4 - 5b(x)c(x) = 6$$. Hence, matrix $A = \begin{bmatrix} 2 & 5b(x) \\ 5c(x) & 7 \end{bmatrix}$ or $\begin{bmatrix} 7 & 5b(x) \\ 5c(x) & 2 \end{bmatrix}$, where $b(x)$ and $c(x)$ are polynomials in $\mathbb{Z}_{10}[x]$. Since $ad - bc = 6$, so $4 - 5b(x)c(x) = 6$.

**References:**


