ON \( W_7 \)- CURVATURE TENSOR OF GENERALIZED SASAKIAN-SPACE-FORMS

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Abstract. The object of the present paper is to study generalized Sasakian-space-forms satisfying certain curvature conditions on \( W_7 \)- curvature tensor. In this paper, we study \( W_7 \)- semisymmetric, \( \xi - W_7 \)- flat, generalized Sasakian-space-forms satisfying \( G.S = 0 \), \( W_7 \)- flat. Also satisfying \( G.P = 0 \), \( G.C = 0 \), \( G.R = 0 \).

1. Introduction

In 2011, M.M. Tripathi and P. Gupta [8] introduced and explored \( \tau \)- curvature tensor in semi-Riemannian manifolds. They gave properties and some identities of \( \tau \)- curvature tensor. They defined \( W_7 \)- curvature tensor of type \((0,4)\) for \((2n + 1)\)-dimensional Riemannian manifold, as

\[
W_7(X, Y, Z, U) = R(X, Y, Z, U) - \frac{1}{2n}\{S(Y, Z)g(X, U) - g(Y, Z)S(X, U)\}
\]

where \( R \) and \( S \) denote the Riemannian curvature tensor of type \((0, 4)\) defined by \( R(X, Y, Z, U) = g(R(X, Y)Z, U) \) and the Ricci tensor of type \((0, 2)\) respectively. The curvature tensor defined by (1.1) is known as \( W_7 \)- curvature tensor. A manifold whose \( W_7 \)- curvature tensor vanishes at every point of the manifold is called \( W_7 \)- flat manifold. They also defined \( \tau \)-conservative semi-Riemannian manifolds and gave necessary and sufficient condition for semi-Riemannian manifolds to be \( \tau \)- conservative.


In differential geometry, the curvature of a Riemannian manifold \((M, g)\) plays a fundamental role as well known, the sectional curvature of a manifold determine the curvature tensor \( R \)- completely. A Riemannian manifold with constant sectional curvature \( c \) is called a real-space form and its curvature tensor is given by the equation

\[
R(X, Y)Z = c\{g(Y, Z)X - g(X, Z)Y\}
\]

for any vector fields \( X, Y, Z \) on \( M \). Models for these spaces are the Euclidean space \( (c = 0) \), the sphere \( (c > 0) \) and the Hyperbolic space \( (c < 0) \).

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A similar situation can be found in the study of complex manifolds from a Riemannian point of view. If $(M, J, g)$ is a Kaehler manifold with constant holomorphic sectional curvature $K(X \wedge JX) = c$, then it is said to be a complex space form and it is well known that its curvature tensor satisfies the equation

$$R(X, Y)Z = \frac{c}{4}(g(Y, Z)X - g(X, Z)Y + g(X, JZ)JY - g(Y, JZ)JX + 2g(X, JY)JZ)$$

for any vector fields $X, Y, Z$ on $M$. These models are $C^n$, $CP^n$ and $CH^n$ depending on $c = 0, c > 0$ and $c < 0$ respectively.

On the other hand, Sasakian-space-forms play a similar role in contact metric geometry. For such a manifold, the curvature tensor is given by

$$R(X, Y)Z = (\frac{c+3}{4})g(Y, Z)X - g(X, Z)Y$$

$$\quad + (\frac{c-1}{4})g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z$$

$$\quad + \eta(X)\eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi$$

for any vector fields $X, Y, Z$ on $M$. These spaces can also be modeled depending on cases $c > -3, c = -3$ and $c < -3$. It is known that any three-dimensional $(\alpha, \beta)$-trans Sasakian manifold with $\alpha, \beta$ depending on $\xi$ is a generalized Sasakian-space-forms [9]. Alegre et al. give results in [11] about B.Y. Chen’s inequality on submanifolds of generalized complex space-forms and generalized Sasakian-space-forms. Al. Ghefari et al. analyse the CR submanifolds of generalized Sasakian-space-forms [12, 13]. Sreenivasa. G.T. Venkatesha and Bagewadi C.S. [14] have some results on $(LCS)_{2n+1}$-Manifolds. S. K. Yadav, P.K. Dwivedi and D. Suthar [15] studied $(LCS)_{2n+1}$- Manifolds satisfying certain conditions on the concircular curvature tensor. De and Sarakar [16] have studied generalized Sasakian-space-forms regarding projecive curvature tensor. Motivated by the above studies, in the present paper, we study flatness and symmetric property of generalized Sasakian-space-forms regarding $W_7 - curvature$ tensor. The present paper is organized as follows:

In this paper, we study the $W_7 - curvature$ tensor of generalized Sasakian-space-forms with certain conditions. In section 2, some preliminary results are recalled. In section 3, we study $W_7$ semisymmetric generalized Sasakian-space-forms. Section 4 deals with $\xi - W_7$ flat generalized Sasakian-space-forms. Generalized Sasakian-space-forms satisfying $G.S = 0$ are studied in section 5. In section 6, $W_7 - flat$ generalized Sasakian-space-forms are studied. Section 7 is devoted to study of generalized Sasakian-space-forms satisfying $G.P = 0$. In section 8 contains generalized Sasakian-space-forms satisfying $G.C = 0$. The last section contains generalized Sasakian-space-forms satisfying $G.R = 0$.

2. Preliminary

An odd - dimensional differentiable manifold $M^{2n+1}$ of differentiability class $C^{r+1}$, there exists a vector valued real linear function $\Phi$, a 1-form $\eta$, associated vector field $\xi$ and the Riemannian metric $g$ satisfying

$$\Phi^2(X) = -X + \eta(X)\xi, \Phi(\xi) = 0$$
(2.2) \[
\eta(\xi) = 1, g(X, \xi) = \eta(X), \eta(\Phi X) = 0
\]

(2.3) \[
g(\Phi X, \Phi Y) = g(X, Y) - \eta(X)\eta(Y)
\]

for arbitrary vector fields \(X\) and \(Y\), then \((M^{2n+1}, g)\) is said to be an almost contact metric manifold [5], and the structure \((\Phi, \xi, \eta, g)\) is called an almost contact metric structure to \(M^{2n+1}\). In view of (2.1), (2.2) and (2.3), we have

(2.4) \[
g(\Phi X, Y) = -g(X, \Phi Y), g(\Phi X, X) = 0
\]

(2.5) \[
\nabla X \eta(Y) = g(\nabla X \xi, Y)
\]

Again we know [10] that in a \((2n+1)\)-dimensional generalized Sasakian-space-forms, we have

\[
R(X, Y)Z = f_1\{g(Y, Z)X - g(X, Z)Y \} + f_2\{g(X, \Phi Z)\Phi Y - g(Y, \Phi Z)\Phi X + 2g(X, \Phi Y)\Phi Z \} + f_3(\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi)
\]

for any vector field \(X, Y, Z\) on \(M^{2n+1}\), where \(R\) denotes the curvature tensor of \(M^{2n+1}\) and \(f_1, f_2, f_3\) are smooth functions on the manifold.

The Ricci tensor \(S\) and the scalar curvature \(r\) of the manifold of dimension \((2n + 1)\) are respectively, given by

(2.7) \[
S(X, Y) = (2nf_1 + 3f_2 - f_3)g(X, Y) - (3f_2 + (2n - 1)f_3)\eta(X)\eta(Y)
\]

(2.8) \[
QX = (2nf_1 + 3f_2 - f_3)X - (3f_2 + (2n - 1)f_3)\eta(X)\xi
\]

(2.9) \[
r = 2n(2n + 1)f_1 + 6nf_2 - 4nf_3
\]

Also for a generalized Sasakian-space-forms, we have

(2.10) \[
R(X, Y)\xi = (f_1 - f_3)\{\eta(Y)X - \eta(X)Y \}
\]

(2.11) \[
R(\xi, X)Y = -R(X, \xi)Y = (f_1 - f_3)\{g(X, Y)\xi - \eta(Y)X \}
\]

(2.12) \[
\eta(R(X, Y)Z) = (f_1 - f_3)\{g(Y, Z)\eta(X) - g(X, Z)\eta(Y) \}
\]

(2.13) \[
S(X, \xi) = 2n(f_1 - f_3)\eta(X)
\]

(2.14) \[
Q\xi = 2n(f_1 - f_3)\xi
\]
where \( Q \) is the Ricci Operator, i.e.

\[
g(QX, Y) = S(X, Y)
\]

(2.15)

For a \((2n+1)\)-dimensional \((n > 1)\) Almost Contact Metric, the \(W_7\)-curvature tensor \( G \) is given by

\[
G(X, Y)Z = R(X, Y)Z - \frac{1}{2n}\{S(Y, Z)X - g(Y, Z)QX\}
\]

(2.16)

the \(W_7\)-curvature tensor \( G \) in a generalized Sasakia-space-forms satisfies

\[
G(X, Y)\xi = (f_1 - f_3)[\eta(Y)X - \eta(X)Y] - \frac{1}{2n}(3f_2 + (2n - 1)f_3)[\eta(X)\eta(Y)\xi - \eta(Y)X]
\]

(2.17)

\[
G(X, \xi)\xi = \frac{1}{2n}(2nf_1 + 3f_2 - f_3)\{X - \eta(X)\xi\}
\]

(2.18)

\[
G(\xi, Y)\xi = (f_1 - f_3)\{\eta(Y)\xi - Y\}
\]

(2.19)

\[
G(\xi, X)Y = (f_1 - f_3)\{2g(X, Y)\xi - \eta(Y)X\} - \frac{1}{2n}S(X, Y)\xi
\]

(2.20)

\[
\eta(G(X, Y)Z) = (f_1 - f_3)\{\eta(X)g(Y, Z) - \eta(Y)g(X, Z)\}\]

(2.21)

\[-\frac{1}{2n}(3f_2 + (2n - 1)f_3)\{g(Y, Z)\eta(X) - \eta(X)\eta(Y)\eta(Z)\}\]

Given an \((2n + 1)\)-dimensional Riemannian manifold \((M, g)\), the Concircular curvature tensor \( \tilde{C} \) is given by

\[
\tilde{C}(X, Y)Z = R(X, Y)Z - \frac{r}{2n(2n + 1)}\{g(Y, Z)X - g(X, Z)Y\}
\]

(2.22)

\[
\tilde{C}(\xi, X)Y = [f_1 - f_3 - \frac{r}{2n(2n + 1)}]\{g(X, Y)\xi - \eta(Y)X\}
\]

(2.23)

and

\[
\eta(\tilde{C}(X, Y)Z) = [f_1 - f_3 - \frac{r}{2n(2n + 1)}]\{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\}
\]

(2.24)

and Projective curvature tensor is given by

\[
P(X, Y)Z = R(X, Y)Z - \frac{1}{2n}[S(Y, Z)X - S(X, Z)Y]
\]

(2.25)

and related term

\[
\eta(P(X, Y)\xi) = 0
\]

(2.26)

\[
\eta(P(X, \xi)Z) = \frac{1}{2n}S(X, Z) - (f_1 - f_3)g(X, Z)
\]

(2.27)

\[
\eta(P(\xi, Y)Z) = (f_1 - f_3)g(Y, Z) - \frac{1}{2n}S(Y, Z)
\]

(2.28)
for any vector field \( X, Y, Z \) on \( M \).

3. \( W_7 \)– Semisymmetric Generalized Sasakian-Space-Forms

**Definition 1.** A \( (2n+1) \)– dimensional \( (n > 1) \) generalized Sasakian-space-forms is said to be \( W_7 \)– semisymmetric if it satisfies \( RG = 0 \), where \( R \) is the Riemannian curvature tensor and \( G \) is the \( W_7 \)– curvature tensor of the space form.

**Theorem 1.** A \( (2n+1) \)– dimensional \( (n > 1) \) generalized Sasakian-space-form is \( W_7 \)– semisymmetric if and only if \( f_1 = f_3 \).

**Proof.** Let us suppose that the generalized Sasakian-space-forms \( M^{2n+1}(f_1, f_2, f_3) \) is \( W_7 \)– semisymmetric, then we have

\[
R(\xi, U)G(X, Y)\xi = 0
\]

The above equation can be written as

\[
R(\xi, U)G(X, Y)\xi - G(R(\xi, U)X, Y)\xi - G(X, R(\xi, U)Y)\xi - G(X, Y)R(\xi, U)\xi = 0
\]

In view of (2.2), (2.10) & (2.11) the above equation reduces to

\[
(f_1 - f_3)\{g(U, G(X, Y))\xi - \eta(G(X, Y))U - g(U, X)G(\xi, Y)\xi + \eta(X)G(U, Y)\xi - g(U, Y)G(X, \xi) + \eta(Y)G(X, U)\xi - \eta(U)G(X, Y)\xi + G(X, Y)U\} = 0
\]

In view of (2.16), (2.17) & (2.18) and taking the inner product of above equation with \( \xi \), we get

\[
(f_1 - f_3)\{g(U, G(X, Y))\xi + g(G(X, Y)U, \xi)\} = 0
\]

\[
(f_1 - f_3)\{g(U, G(X, Y))\xi + \eta(G(X, Y)U)\} = 0
\]

This implies either \( f_1 = f_3 \) or

\[
g(U, G(X, Y))\xi + \eta(G(X, Y)U) = 0
\]

In the light of equation (2.17) and (2.21), the above equation gives

\[
\eta(Y)g(X, U) - \eta(X)g(U, Y) = 0
\]

which is not possible in generalized Sasakian-space-form. Conversely, if \( f_1 = f_3 \), then from (2.11), \( R(\xi, U) = 0 \). Then obviously \( RG = 0 \) is satisfied. This completes the proof. \( \square \)

4. \( \xi - W_7 \)– Flat Generalized Sasakian-Space-Forms

**Definition 2.** A \( (2n+1) \)– dimensional \( (n > 1) \) generalized Sasakian-space-form is said to be \( W_7 \)– flat \([6]\) if \( G(X, Y)\xi = 0 \) for all \( X, Y \in TM \).

**Theorem 2.** A \( (2n+1) \)– dimensional \( (n > 1) \) generalized Sasakian-space-form is \( \xi - W_7 \)– flat if and only if it is \( \eta \)– Einstein Manifold.
Proof. Let us consider that a generalized Sasakian-space-forms is $\xi - W_7$- flat, i.e. $G(X,Y)\xi = 0$. Then from (2.16), we have

\begin{align}
R(X,Y)\xi &= \frac{1}{2n}\{S(Y,\xi)X - g(Y,\xi)QX\} \\
R(X,Y)\xi &= \frac{1}{2n}\{S(Y,\xi)X - \eta(Y)QX\}
\end{align}

By using (2.10) & (2.12) above equation becomes

\begin{align}
(f_1 - f_3)\{\eta(Y)X - \eta(X)Y\} &= \frac{1}{2n}\{2n(f_1 - f_3)\eta(Y)X - \eta(Y)QX\}
\end{align}

On solving, we get

\begin{align}
\eta(Y)QX = 2n(f_1 - f_3)\eta(X)Y \\
QX = 2n(f_1 - f_3)\eta(X)\xi
\end{align}

Now, taking the inner product of the above equation with $U$, we get

\begin{align}
S(X,U) = 2n(f_1 - f_3)\eta(X)\eta(U)
\end{align}

which implies generalised Sasakian-space-forms is an $\eta$- Einstein Manifold. Conversely, suppose that (4.6) is satisfied. Then from (4.1) & (4.4), we get

$$G(X,Y)\xi = 0$$

This completes the proof.\hfill \Box

5. Generalized Sasakian-Space-Forms Satisfying $G.S = 0$

**Theorem 3.** A generalized Sasakian-space-form $M^{2n+1}(f_1, f_2, f_3)$ satisfies the condition $G(\xi,X).S = 0$ if and only if either $M^{2n+1}(f_1, f_2, f_3)$ has $f_1 = f_3$ or an Einstein Manifold.

**Proof.** The condition $G(\xi,X).S = 0$ implies that

$$S(G(\xi,X)Y,Z) + S(Y,G(\xi,X)Z) = 0$$

for any vector fields $X,Y,Z$ on $M$. Substituting (2.18) in above equation, we obtain

\begin{align}
(f_1 - f_3)\{2n(f_1 - f_3)g(X,Y)\eta(Z) - \eta(Y)S(X,Z) \\
+2n(f_1 - f_3)g(X,Y)\eta(Z)\} - \frac{1}{2n}\{2n(f_1 - f_3)S(X,Y)\eta(Z) \\
+(f_1 - f_3)\{2n(f_1 - f_3)g(X,Z)\eta(Y) - S(Y,X)\eta(Z) \\
+2n(f_1 - f_3)g(X,Z)\eta(Y)\} - \frac{1}{2n}\{2n(f_1 - f_3)S(X,Z)\eta(Y) \\
= 0
\end{align}
For \( Z = \xi \), the last equation is equivalent to
\[
(5.2) \quad (f_1 - f_3)\{2n(f_1 - f_3)g(X, Y) - 2n(f_1 - f_3)\eta(Y)\eta(Z)
+ 2n(f_1 - f_3)g(X, Y)\} - (f_1 - f_3)S(X, Y)
+ (f_1 - f_3)(2n(f_1 - f_3)\eta(X)\eta(Y) - S(Y, X)
+ 2n(f_1 - f_3)\eta(X)\eta(Y)) - 2n(f_1 - f_3)(f_1 - f_3)\eta(X)\eta(Y)
= 0
\]
Using (2.12), we obtain
\[
(5.3) \quad S(X, Y) = 2n(f_1 - f_3)g(X, Y)
S(X, Y) = \lambda g(X, Y)
\]
which implies, it is an Einstein Manifold where \( \lambda = 2n(f_1 - f_3) \). \( \square \)

6. \( W_7 \)- flat Generalized Sasakian-space-forms

**Theorem 4.** A \((2n + 1)-\) dimensional \((n > 1)\) generalized Sasakian-space-form is \( W_7 \)- flat if and only if \( f_1 = \frac{3f_2}{(1-2n)} = f_3 \).

**Proof.** For a \((2n + 1)-\) dimensional \( W_7 \)- flat generalized Sasakian-space-forms, we have from (2.16)

\[
(6.1) \quad R(X, Y)Z = \frac{1}{2n} \{ S(Y, Z)X - g(Y, Z)QX \}
\]

In view of (2.7) & (2.8), the above equation takes the form

\[
(6.2) \quad R(X, Y)Z = \frac{1}{2n} \{ -(3f_2 + (2n - 1)f_3)(\eta(Y)\eta(Z)X + g(Y, Z)\eta(X)\xi) \}
\]

By virtue of (2.6), the above equation reduces to

\[
(6.3) \quad f_1 \{ g(Y, Z)X - g(X, Z)Y \}
+ f_2 \{ g(X, \Phi Z)\Phi Y - g(Y, \Phi Z)\Phi X + 2g(X, \Phi Y)\Phi Z \}
+ f_3 \{ \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi \}
= \frac{1}{2n} \{ -(3f_2 + (2n - 1)f_3)(\eta(Y)\eta(Z)X + g(Y, Z)\eta(X)\xi) \}
\]

Now, replacing \( Z \) by \( \Phi Z \) in the above equation, we obtain

\[
(6.4) \quad f_1 \{ g(Y, \Phi Z)X - g(X, \Phi Z)Y \}
+ f_2 \{ g(X, \Phi^2 Z)\Phi Y - g(Y, \Phi^2 Z)\Phi X + 2g(X, \Phi Y)\Phi^2 Z \}
+ f_3 \{ g(X, \Phi Z)\eta(Y)\xi - g(Y, \Phi Z)\eta(X)\xi \}
= \frac{1}{2n} \{ -(3f_2 + (2n - 1)f_3)g(Y, \Phi Z)\eta(X)\xi) \}
\]

Taking inner product of above equation with \( \xi \), we get
In view of (2.1) & (2.2), we obtain

\[(6.6) \quad (2nf_1 + 3f_2 - f_3)g(Y, \Phi Z)\eta(Y) - 2n(f_1 - f_3)g(X, \Phi Z)\eta(Y) = 0\]

Putting \(X = \xi\) in above equation, we get

\[(6.7) \quad (2nf_1 + 3f_2 - f_3)g(Y, \Phi Z) = 0\]

Since \(g(Y, \Phi Z) \neq 0\) in general, we obtain

\[(6.8) \quad 2nf_1 + 3f_2 - f_3 = 0\]

Again replacing \(X\) by \(\Phi X\) in equation (6.3), we get

\[(6.9) \quad f_1 \{g(Y, Z)\Phi X - g(\Phi X, Z)Y\} + f_2 \{g(\Phi X, \Phi Z)\Phi Y - g(Y, \Phi Z)\Phi^2 X + 2g(\Phi X, \Phi Y)\Phi Z\} + f_3 \{\eta(\Phi X)\eta(Y)\eta(Y)\Phi X + g(\Phi X, Z)\eta(Y)\xi - g(Y, Z)\eta(\Phi X)\xi\} = \frac{1}{2n} \{-(3f_2 + (2n - 1)f_3)g(Y, \Phi Z)\eta(Y)\} \]

Taking inner product with \(\xi\)

\[(6.10) \quad (f_1 - f_3)g(\Phi X, Z)\eta(Y) = 0\]

putting \(Y = \xi\), we get

\[(6.11) \quad (f_1 - f_3)g(\Phi X, Z) = 0\]

Since \(g(\Phi X, Z) \neq 0\) in general, we obtain

\[(6.12) \quad f_3 = f_1\]

From equation (6.8) and (6.12), we get

\[(6.13) \quad f_1 = \frac{3f_2}{1 - 2n} = f_3\]

Conversely, suppose that \(f_1 = \frac{3f_2}{1 - 2n} = f_3\) satisfies in generalized Sasakian-spaceforms and then we have

\[(6.14) \quad S(X, Y) = 0\]

\[(6.15) \quad QX = 0\]

Also, in view of (2.16), we have

\[(6.16) \quad G(X, Y, Z, U) = \mathcal{R}(X, Y, Z, U)\]
where \( G(X, Y, Z, U) = g(G(X, Y)Z, U) \) and \( R(X, Y, Z, U) = g(R(X, Y)Z, U) \).

Putting \( Y = Z = e_i \) in above equation and taking summation over \( i, 1 \leq i \leq 2n+1 \), we get

\[
(6.17) \quad \sum_{i=1}^{2n+1} G(X, e_i, e_i, U) = \sum_{i=1}^{2n+1} R(X, e_i, e_i, U) = S(X, U)
\]

In view of (2.6) & (6.16), we have

\[
G(X, Y, Z, U) = f_1\{g(Y, Z)g(X, U) - g(X, Z)g(Y, U)\}
+ f_2\{g(X, \Phi Z)g(\Phi Y, U) - g(Y, \Phi Z)g(\Phi X, U) + 2g(X, \Phi Y)g(\Phi Z, U)\}
+ f_3\{\eta(X)\eta(Z)g(Y, U) - \eta(Y)\eta(Z)g(X, U)\}
+ g(X, Z)\eta(Y)\eta(U) - g(Y, Z)\eta(X)\eta(U)\}
(6.18)
\]

Now, putting \( Y = Z = e_i \) in above equation and taking summation over \( i, 1 \leq i \leq 2n+1 \), we get

\[
(6.19) \quad \sum_{i=1}^{2n+1} G(X, e_i, e_i, U) = 2nf_1g(X, U) + 3f_2g(\Phi X, \Phi U) - f_3\{(2n-1)\eta(X)\eta(U) + g(X, U)\}
\]

In view of (6.14) & (6.17), we have

\[
(6.20) \quad 2nf_1g(X, U) + 3f_2g(\Phi X, \Phi U) - f_3\{(2n-1)\eta(X)\eta(U) + g(X, U)\} = 0
\]

Putting \( X = U = e_i \) in above equation and taking summation over \( i, 1 \leq i \leq 2n+1 \), we get \( f_1 = 0 \). Then in view of (6.12), \( f_2 = f_3 = 0 \). Therefore, we obtain from (2.6)

\[
(6.21) \quad R(X, Y)Z = 0
\]

Hence in view of (6.14), (6.15) & (6.21), we have \( G(X, Y)Z = 0 \). This completes the proof. \( \square \)

7. Generalized Sasakian-space-forms satisfying \( G.P = 0 \)

**Theorem 5.** A generalized Sasakian-space-form \( M^{2n+1}(f_1, f_2, f_3) \) satisfies the condition

\[ G(\xi, X).P = 0 \]

if and only if \( M^{2n+1}(f_1, f_2, f_3) \) has either the sectional curvature \( (f_1 - f_3) \) or the function \( f_1, f_2 \) and \( f_3 \) are linearly dependent such that \( 2nf_1 - 3f_2 + (1-4n)f_3 = 0 \).

**Proof.** The condition \( G(\xi, X).P = 0 \) implies that

\[ (G(\xi, X)P)(Y, Z, U) = G(\xi, X)P(Y, Z)U - P(G(\xi, X)Y, Z)U - P(Y, G(\xi, X)Z)U - P(Y, Z)G(\xi, X)U = 0 \]

for any vector fields \( X, Y, Z \) on \( M \).

In view of (2.7), we obtain from (2.25)

\[
(7.2) \quad \eta(P(X, Y)Z) = 0
\]
Since,
\begin{equation}
(3.3)
G(\xi, X)P(Y, Z)U = (f_1 - f_3)\{2g(X, P(Y, Z)U)\xi - \eta(P(Y, Z)U)X\} - \frac{1}{2n}S(X, P(Y, Z)U)\xi
\end{equation}

\begin{equation}
(4.4)
P(G(\xi, X)Y, Z)U = (f_1 - f_3)\{2g(X, Y)P(\xi, Z)U - \eta(Y)P(X, Z)U\} - \frac{1}{2n}S(X, Y)P(\xi, Z)U
\end{equation}

Finally, we conclude that
\begin{equation}
(5.5)
P(Y, Z)G(\xi, X)U = (f_1 - f_3)\{2g(X, U)P(Y, Z)\xi - \eta(U)P(Y, Z)X\} - \frac{1}{2n}S(X, U)P(Y, Z)\xi
\end{equation}

So, substituting (3.3), (4.4) and (5.5) in (7.1), we get
\begin{equation}
(6.6)
(f_1 - f_3)\{2g(X, P(Y, Z)U)\xi - \eta(P(Y, Z)U)X - 2g(X, Y)P(\xi, Z)U
\end{equation}

\begin{equation}
+ \eta(Y)P(X, Z)U - 2g(X, Z)P(Y, \xi)U + \eta(Z)P(Y, X)U
\end{equation}

\begin{equation}
- 2g(X, U)P(Y, Z)\xi + \eta(U)P(Y, Z)X\} - \frac{1}{2n}\{S(X, P(Y, Z)U)\xi
\end{equation}

\begin{equation}
- S(X, Y)P(\xi, Z)U - S(X, Z)P(Y, \xi)U - S(X, U)P(Y, Z)\xi
\end{equation}

\begin{equation}
= 0
\end{equation}

Taking inner product with \(\xi\)
\begin{equation}
2(f_1 - f_3)\{g(X, R(Y, Z)U) - (f_1 - f_3)(g(X, Y)g(Z, U) - g(X, Z)g(Y, U)))
\end{equation}

\begin{equation}
- \frac{1}{2n}\{S(X, R(Y, Z)U) - (f_1 - f_3)(S(X, Y)g(Z, U) - S(X, Z)g(Y, U))\}
\end{equation}

\begin{equation}
= 0
\end{equation}

Simplifying above equation, we get
\begin{equation}
(7.7)
(2n f_1 - 3 f_2 + (1 - 4n) f_3)\{g(X, R(Y, Z)U) - (f_1 - f_3)(g(X, Y)g(Z, U) - g(X, Z)g(Y, U))\} = 0
\end{equation}

which say us \(M^{2n+1}(f_1, f_2, f_3)\) has the sectional curvature \((f_1 - f_3)\) or the functions \(f_1, f_2\) and \(f_3\) are linearly dependent such that \((2n f_1 - 3 f_2 + (1 - 4n) f_3) = 0\). \[\Box\]

8. Generalized Sasakian-space-forms satisfying \(G\hat{C} = 0\)

**Theorem 6.** A generalized Sasakian-space-form \(M^{2n+1}(f_1, f_2, f_3)\) satisfies the condition

\begin{equation}
G(\xi, X)\hat{C} = 0
\end{equation}

if and only if either the scalar curvature \(\tau\) of \(M^{2n+1}(f_1, f_2, f_3)\) is \(\tau = (f_1 - f_3)2n(2n + 1)\) or the functions \(f_2\) and \(f_3\) are linearly dependent such that \(3 f_2 + (2n - 1) f_3 = 0\).

**Proof.** The condition \(G(\xi, X)\hat{C} = 0\) implies that
\begin{equation}
(8.1) \ (G(\xi, X)\hat{C})(Y, Z, U) = G(\xi, X)\hat{C}(Y, Z)U - \hat{C}(G(\xi, X)Y, Z)U
\end{equation}

\begin{equation}
- \hat{C}(Y, G(\xi, X)Z)U - \hat{C}(Y, Z)G(\xi, X)U = 0
\end{equation}

for any vector fields \(X, Y, Z\) and \(U\) on \(M\). From (2.22) and (2.23), we can easily to see that
(8.2) 
\[ G(\xi, X)\tilde{C}(Y, Z)U = (f_1 - f_3)\{2g(X, \tilde{C}(Y, Z)U)\xi - \eta(\tilde{C}(Y, Z)U)X\} - \frac{1}{2n}S(X, \tilde{C}(Y, Z)U)\xi \]

(8.3) 
\[ \tilde{C}(G(\xi, X)Y, Z)U = (f_1 - f_3)\{2g(Y, \tilde{C}(\xi, Z)U) - \eta(\tilde{C}(Y, Z)U)\} - \frac{1}{2n}S(X, Y)\tilde{C}(\xi, Z)U \]

(8.4) 
\[ \tilde{C}(Y, G(\xi, X)Z)U = (f_1 - f_3)\{2g(Z, \tilde{C}(Y, \xi)U - \eta(Z)\tilde{C}(Y, X)U\} - \frac{1}{2n}S(X, Z)\tilde{C}(Y, \xi)U \]

and

(8.5) 
\[ \tilde{C}(Y, Z)G(\xi, X)U = (f_1 - f_3)\{2g(U, \tilde{C}(Y, Z)\xi - \eta(U)\tilde{C}(Y, Z)X\} - \frac{1}{2n}S(X, U)\tilde{C}(Y, Z)\xi \]

Thus, substituting (8.2), (8.3), (8.4) and (8.5) in (8.1) and after from necessary abbreviations, (8.1) takes from

\[ (2nf_1 - 3f_2 - (4n - 1)f_3)\{g(X, R(Y, Z)U) - (f_1 - f_3)(g(Z, U)g(X, Y) - g(X, Z)g(Y, U)) \]
\[ + (f_1 - f_3 - \frac{\tau}{2n(2n+1)})(3f_2 + (2n - 1)f_3)\{g(X, Z)\eta(Y)\eta(U) - g(X, Y)\eta(Z)\eta(U) \} = 0 \]

Now putting \( U = \xi \) in the above equation, we get

\[ (f_1 - f_3 - \frac{\tau}{2n(2n+1)})(3f_2 + (2n - 1)f_3)\{g(X, Z)\eta(Y) - g(X, Y)\eta(Z) \} = 0 \]

This equation tells us that either \( M^{2n+1}(f_1, f_2, f_3) \) has either the scalar curvature \( \tau = (f_1 - f_3)2n(2n+1) \) or the functions \( f_2 \) and \( f_3 \) are linearly dependent such that 
\[ 3f_2 + (2n - 1)f_3 = 0. \]

\[ \square \]

9. Generalized Sasakian-space-forms satisfying \( G.R = 0 \)

**Theorem 7.** A \((2n + 1)-\)dimensional \((n > 1)\) generalized Sasakian-space-form satisfying \( G.R = 0 \) is an \( \eta \)-Einstein Manifold.

**Proof.** The condition \( G(\xi, X).R = 0 \) yields to

(9.1) 
\[ G(\xi, X)R(Y, Z)U - R(G(\xi, X)Y, Z)U - R(Y, G(\xi, X)Z)U - R(Y, Z)G(\xi, X)U = 0 \]

for any vector fields \( X, Y, Z, U \) on \( M \). In view of (2.20), we obtain

(9.2) 
\[ G(\xi, X)R(Y, Z)U = (f_1 - f_3)\{2g(X, R(Y, Z)U)\xi - \eta(R(Y, Z)U)X\} \]

\[ - \frac{1}{2n}S(X, R(Y, Z)U)\xi \]

On the other hand, by direct calculations, we have

(9.3) 
\[ R(G(\xi, X)Y, Z)U = (f_1 - f_3)\{2g(X, Y)R(\xi, Z)U - \eta(Y)R(X, Z)U\} \]

\[ - \frac{1}{2n}S(X, Y)R(\xi, Z)U \]

and

(9.4) 
\[ R(Y, G(\xi, X)Z)U = (f_1 - f_3)\{2g(Z, Y)R(Y, \xi)U - \eta(Z)R(Y, X)U\} \]

\[ - \frac{1}{2n}S(X, Z)R(Y, \xi)U \]
and
\[ R(Y, Z)G(\xi, X)U = (f_1 - f_3)(2g(X, U)R(Y, Z)\xi - \eta(U)R(Y, Z)X) \]
(9.5)
\[ -\frac{1}{2n}S(X, U)R(Y, Z)\xi \]

Substituting (9.2), (9.3), (9.4) & (9.5) in (9.1), we get
\[ (f_1 - f_3)(2g(X, R(Y, Z)U)\xi - (f_1 - f_3)(g(Z, U)\eta(Y)X - g(Y, U)\eta(Z)X) \]
\[ -2g(X, Y)(f_1 - f_3)(g(Z, U)\xi - \eta(U)Z) + \eta(Y)R(X, Z)U - 2g(X, Z)(f_1 - f_3)(\eta(U)Y - g(Y, U)\xi) \]
\[ + (\eta(Z)R(Y, U)U - 2g(X, U)(f_1 - f_3)(\eta(Z)Y - \eta(Y)Z) + \eta(U)R(Y, Z)X \]
\[ + \frac{1}{2n}\{-S(X, R(Y, Z)U)\xi + S(X, Y)(f_1 - f_3)(g(Z, U)\xi - \eta(U)Z) \]
\[ + S(X, Z)(f_1 - f_3)(\eta(U)Y - g(Y, U)\xi) + S(X, U)(f_1 - f_3)(\eta(Z)Y - \eta(Y)Z)\} = 0 \]

Taking inner product with \( \xi \), above equation implies that
\[ (f_1 - f_3)(2g(X, R(Y, Z)U) - 2(f_1 - f_3)g(X, Y)g(Z, U) + (f_1 - f_3)g(X, Y)\eta(Z)\eta(U) \]
\[ - (f_1 - f_3)g(X, Z)\eta(Y)\eta(U) + 2(f_1 - f_3)g(X, Z)g(Y, U) \]
\[ + \frac{1}{2n}\{-S(X, R(Y, Z)U) + (f_1 - f_3)S(X, Y)(g(Z, U) \]
\[ - \eta(Z)\eta(U)) + (f_1 - f_3)S(X, Z)(\eta(U)\eta(Y) - g(Y, U))\} = 0 \]

In consequence of (2.6), (2.10), (2.11) and (2.12) the above equation takes the form
\[ (f_1 - f_3)(2f_3g(X, Y)g(Z, U) - 2f_3g(X, Z)g(Y, U) + 2f_2g(X, \Phi Z)g(Y, \Phi U) \]
\[ - 2f_2g(Z, \Phi U)g(X, \Phi Y) + 4f_2g(X, \Phi U)g(Y, \Phi Z) + (3f_3 - f_1)g(X, Z)\eta(Y)\eta(U) \]
\[ + (f_1 - f_3)g(X, Y)\eta(Z)\eta(U) + 2f_3g(Y, U)\eta(X)\eta(Z) - 2f_3g(Z, U)\eta(X)\eta(Y) \]
\[ + \frac{1}{2n}\{-f_3S(X, Y)g(Z, U) + f_3S(X, Z)g(Y, U) - f_2S(X, \Phi Z)g(Y, \Phi U) \]
\[ + f_2S(X, \Phi Y)g(Z, \Phi U) - 2f_2g(Y, \Phi Z)S(X, \Phi U) + (f_1 - f_3)S(X, Z)\eta(Y)\eta(U) \]
\[ + (2f_3 - f_1)S(X, Y)\eta(Z)\eta(U) - f_3g(Y, U)S(X, \xi)\eta(Z) + f_3g(Z, U)S(X, \xi)\eta(Y))\} = 0 \]

Putting \( Z = U = e_i \), in the above equation and taking summation over \( i, 1 \leq i \leq 2n + 1 \), we get
\[ S(X, Y) = \frac{2n(f_1 - f_3)(f_1 + 6f_2 + (4n - 3)f_3)(f_1 + 3f_2 + 2(n - 2)f_3)}{(f_1 + 3f_2 + 2(n - 2)f_3)^2}\cdot g(X, Y) - \frac{2n(f_1 - f_3)(3(2n + 1)f_3 + e)}{(f_1 + 3f_2 + 2(n - 2)f_3)^2}\cdot \eta(X)\eta(Y) \]

which implies that
\[ S(X, Y) = \lambda_1g(X, Y) - \lambda_2\eta(X)\eta(Y) \]

which show that \( M^{2n+1} \) is an \( \eta \)-Einstein manifold. \( \square \)

10. References

ON $W_\gamma$– CURVATURE TENSOR OF GENERALIZED SASAKIAN-SPACE-FORMS


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