On the Construction of Weighing Matrices from Coherent Configuration

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Abstract: In this paper we forward two methods of construction of conference matrices of order 6 by suitable combinations of adjacency matrices of suitable coherent configuration.

Key words: Coherent configuration, weighing matrix, conference matrices.

1. Introduction: We begin with the following definition:

1.1. WEIGHING MATRICES: A weighing matrix \( W \) of order \( n \) and weight \( w \) is an \( n \times n \) matrix with entries \((0, \pm 1)\), such that

\[
WW^T = wI_n,
\]

where \( W^T \) is the transpose of \( W \) and \( I_n \), is the identity matrix of order \( n \). A weighing matrix of order \( n \) and weight \( w \) is denoted by \( W(n, w) \).

(i) \( W(n, n) \) is a hadamard matrix.

(ii) \( W(n, n - 1), n \) even with zeros on the diagonal such that \( WW^T = (n - 1)I_n \) is conference matrix.

(iii) If \( n \equiv 2 \mod 4 \), such that \( W = W^T \) is symmetric conference matrix.

(Vide: [1] and [5])

Example:

\[
W(6,5) = \begin{bmatrix}
0 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & -1 & -1 & 1 \\
1 & 1 & 0 & 1 & -1 & -1 \\
1 & -1 & 1 & 0 & 1 & -1 \\
1 & -1 & -1 & 1 & 0 & 1 \\
1 & 1 & -1 & -1 & 1 & 0
\end{bmatrix}
\]

(Vide: [5])

1.2. PROPERTIES OF WEIGHING MATRICES:

If \( W \) is a \( W(n, w) \) then:

(i) The rows of \( W \) are pairwise orthogonal. Similarly, the columns are pairwise orthogonal.

(ii) Each row and each column of \( W \) has exactly \( w \) non-zero elements.

(iii) \( W^T W = wI \), since the definition means that \( W^{-1} = w^{-1} W^T \) where \( W^{-1} \) is the inverse of \( W \).

(iv) If there is a \( W(n, p) \) then there is a symmetric \( W(n^2, p^2) \).

(v) For a weighing matrix \( W(n, n - 1) \) \( WW^T = (n - 1)I_n \), then \( \det W \equiv W(n) = (n - 1)^{n/2} \). (Vide: [1])

1.3. CONFERENCE MATRICES:

A conference matrix of order \( n \) is an \( n \times n \) matrix \( M \) with diagonal entries 0 and other entries \( \pm 1 \), which satisfies \( MM^T = (n - 1)I_n \).

Where : \( M^T \) is transpose of \( M \) and \( I_n \) is the identity matrix.
Examples: \[ M = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & -1 & -1 & 1 \\ 1 & 1 & 0 & 1 & -1 & -1 \\ 1 & -1 & 1 & 0 & 1 & -1 \\ 1 & -1 & -1 & 1 & 0 & 1 \\ 1 & 1 & -1 & -1 & 1 & 0 \end{bmatrix} \]

(vide : [3])

1.4. SYMMETRIC CONFERENCE MATRICES:
A conference matrix \( M \) with entries \( 0, 1, \text{and} -1 \) is called symmetric conference matrix if \( MM^T = MM^T = nI_n \)

Where: \( n \) is order of matrix, \( I_n \) is identity matrix and \( M^T \) is the transpose of \( M \).

(vide : [3]) Examples:

\[
M = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 1 & 1 & -1 \\ 1 & -1 & 0 & -1 & 1 & 1 \\ 1 & 1 & -1 & 0 & 1 & -1 \\ 1 & 1 & 1 & -1 & 0 & 1 \\ 1 & -1 & 1 & 1 & -1 & 0 \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & -1 & -1 & 1 \\ 1 & 1 & 0 & 1 & -1 & -1 \\ 1 & -1 & 1 & 0 & 1 & -1 \\ 1 & -1 & -1 & 1 & 0 & 1 \\ 1 & 1 & -1 & -1 & 1 & 0 \end{bmatrix}
\]

(vide : [3])

1.5. PROPERTIES OF SYMMETRIC CONFERENCE MATRICES AND CONFERENCE MATRICES:
Some important properties of Symmetric Conference matrices and conference matrices are given below:

1. The order of conference matrix is of the form \( 4t + 2 \),
2. \( n - 1 \) where \( n \) is the order of a conference matrix , must be the sum of two squares;
3. If there is a conference matrix of order \( n \) then there is a symmetric conference matrix of order \( n \) with zero diagonal .The two forms are equivalent as one can be transformed into the other by
   (i) Interchanging rows (columns) or
   (ii) multiplying rows (columns) by -1;
4. A conference matrix is said to be normalized if it has first rows and columns all plus ones.
5. \( M^{-1} = nM^T \)

(vide : [3])

1.6. SKEW-CONFERENCE MATRIX:
A conference matrix \( M \) with entries 0, \text{and} \pm 1 \) is called skew symmetric matrix conference matrix if \( M^T = -M \)

Where: \( T \) denotes the matrix transpose.

(vide : [11])
\[ M = \begin{bmatrix} 0 & 1 & 1 & -1 & 1 & -1 \\ -1 & 0 & 1 & 1 & -1 & 1 \\ -1 & -1 & 0 & 1 & 1 & -1 \\ 1 & -1 & -1 & 0 & 1 & 1 \\ -1 & 1 & -1 & 0 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 & 0 \end{bmatrix} \]

**Example:**

1. **COHERENT CONFIGURATION (CC):**

   Let \( X \) be a finite set. A coherent configuration on \( X \) is a set \( C = \{C_1, C_2, C_3, \ldots, C_m\} \) of binary relation on \( X \) (subsets of \( X \times X \)) satisfying the following four conditions:

   (i) 'C' is a partition of \( X \times X \) that is 
   \[ \bigcup_{i=1}^{m} C_i = X \times X \]

   (ii) There exist a sub set \( C_\alpha \) of \( C \) which is a partition of the diagonal \( D = \{(x, x) : x \in X\} \)

   (iii) For every relation \( C_j \in C \), its converse \( C_j' = \{(\beta, \alpha) : (\alpha, \beta) \in C_j\} \) is in \( C \) say \( C_j = C_j' \in C_k \)

   (iv) There exist integer \( P_{ij}^k \) for \( 1 \leq i, j, k \leq m \) such that for any \((\alpha, \beta) \in C_k \) the number of points \( \gamma \in X \) such that \((\alpha, \gamma) \in C_i \) and \((\gamma, \beta) \in C_j \) is equal to \( P_{ij}^k \) (and in particular, is independent of the choice of \((\alpha, \beta) \in C_k \).

   That is we have
   \[ P_{ij}^k = \left| C_i(\alpha) \cap C_j'(\beta) \right| \text{ for } (\alpha, \beta) \in C_k \]
   Where \( C(\alpha) = \{\beta \in X : (\alpha, \beta) \in C\} \).

C.C. is also defined by adjacency matrices of classes of \( C \). If \( A_1, A_2, \ldots, A_m \) are adjacency matrices of \( C_1, C_2, \ldots, C_m \) respectively then the axioms takes the following from

(i) \( A_1 + A_2 + \ldots + A_m = J \)

(ii) There exist a sub set of \( \{A_1, A_2, \ldots, A_m\} \) with sum \( I = \text{identity matrix} \); 

(iii) Each elements of the set \( \{A_1, \ldots, A_m\} \) is closed under transposition ;

(iv) \( A_j A_j = \sum_{i=1}^{m} P_{ij}^k A_k \) where \( P_{ij}^k \) are non-negative integers.

**Vide: Singh and Manjhi [8].**

2. **MAIN WORK:**

   In this paper we construct two conference matrices each of orders 6 by suitable linear combination of coherent configurations.

2.1. **CONSTRUCTION OF SYMMETRIC CONFERENCE MATRIX OF ORDER 6:**

Consider \( X = \{1, 2, 3, 4, 5, 6\} \) and a partition \( C = \{C_1, C_2, C_3, C_4, C_5, C_6\} \) of \( X \times X \) where
\[ C_1 = \{(i, i) : i = 1\}, \]
\[ C_2 = \{(i, i) : i = 2, 3, 4, 5, 6\}, \]
\[ C_3 = \{(i, i) : i = 2, 3, 4, 5, 6\}, \]
\[ C_4 = \{(i, i) : i = 2, 3, 4, 5, 6\}, \]
\[ C_5 = \{(2, i) : i = 3, 6\} \cup \{(3, i) : i = 2, 4\} \cup \{(4, i) : i = 3, 5\} \cup \{(5, i) : i = 4, 6\} \cup \{(6, i) : i = 2, 5\}, \]
\[ C_6 = \{(2, i) : i = 4, 5\} \cup \{(3, i) : i = 5, 6\} \cup \{(4, i) : i = 2, 6\} \cup \{(5, i) : i = 2, 3\} \cup \{(6, i) : i = 3, 4\}. \]

Then adjacency matrices \( M_1, M_2, M_3, M_4, M_5 \) and \( M_6 \) of \( C_1, C_2, C_3, C_4, C_5 \) and \( C_6 \) respectively are given below:

\[
M_1 = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
M_2 = \begin{bmatrix}
0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
M_3 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
M_4 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
M_5 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0
\end{bmatrix}
\]

\[
M_6 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0
\end{bmatrix}
\]

We see that

1. \( M_1 + M_2 + M_3 + M_4 + M_4 + M_1 = J_6 \)
2. \( M_1 + M_4 = I_6 \)
3. \( M'_1 = M_1, M'_2 = M_2, M'_3 = M_3, M'_4 = M_4, M'_5 = M_5, M'_6 = M_6 \)
4. We see the following calculations:

(i) \( M'_1^2 = M_1, M'_1M_2 = M_2, M'_1M_3 = 0, M'_1M_4 = 0, M'_1M_5 = 0, M'_1M_6 = 0 \)
(ii) \( M'_2^2 = 0, M'_2M_3 = 5M_1, M'_2M_4 = M_2, M'_2M_5 = 2M_2, M'_2M_6 = 2M_2 \)
(iii) \( M'_3^2 = 0, M'_3M_4 = 0, M'_3M_5 = 0, M'_3M_6 = 0 \)
(iv) \( M'_4^2 = M_4, M'_4M_5 = M_5, M'_4M_6 = M_6 \)
(v) \( M'_5^2 = 2M_4 + M_6, M'_5M_6 = M_5 + M_6 \)
(vi) \( M'_6^2 = 2M_4 + M_5 \)

Hence product of any two adjacency matrices is some linear combinations of adjacency matrices.

Thus the set \( C = \{C_1, C_2, C_3, C_4, C_5, C_6\} \) is a C.C.
Consider the matrix
\[ M = \begin{bmatrix}
0 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & -1 & -1 & 1 \\
1 & 1 & 0 & 1 & -1 & -1 \\
1 & -1 & 1 & 0 & 1 & -1 \\
1 & -1 & -1 & 1 & 0 & 1 \\
1 & 1 & -1 & -1 & 1 & 0
\end{bmatrix} \]

\[ M = \begin{bmatrix}
0 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & -1 & -1 & 1 \\
1 & 1 & 0 & 1 & -1 & -1 \\
1 & -1 & 1 & 0 & 1 & -1 \\
1 & -1 & -1 & 1 & 0 & 1 \\
1 & 1 & -1 & -1 & 1 & 0
\end{bmatrix} \]

\[ MM^T = \begin{bmatrix}
0 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & -1 & -1 & 1 \\
1 & 1 & 0 & 1 & -1 & -1 \\
1 & -1 & 1 & 0 & 1 & -1 \\
1 & -1 & -1 & 1 & 0 & 1 \\
1 & 1 & -1 & -1 & 1 & 0
\end{bmatrix} \]

\[ = 5I_6 = (6-1)I_6 \]

\[ \Rightarrow MM^T = (6-1)I_6 \]

\[ M^TM = \begin{bmatrix}
0 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & -1 & -1 & 1 \\
1 & 1 & 0 & 1 & -1 & -1 \\
1 & -1 & 1 & 0 & 1 & -1 \\
1 & -1 & -1 & 1 & 0 & 1 \\
1 & 1 & -1 & -1 & 1 & 0
\end{bmatrix} \]

\[ = 5I_6 = (6-1)I_6 \]

\[ \Rightarrow M^TM = (6-1)I_6 \]

Thus \( MM^T = M^TM = (6-1)I_6 \)

Which show that \( M \) is a symmetric conference matrix of order 6.

2.2. Consider \( X = \{1,2,3,4,5,6\} \) and a partition \( C = \{C_1, C_2, C_3, C_4, C_5, C_6\} \) of \( X \times X \) where

\( C_1 = \{(i, i) : i = 1\}, \)

\( C_2 = \{(1, i) : i = 2,3,4,5,6\}, \)

\( C_3 = \{(i, 1) : i = 2,3,4,5,6\}, \)

\( C_4 = \{(i, i) : i = 2,3,4,5,6\} \)

\( C_5 = \{(2, i) : i = 4,5\} \cup \{(3, i) : i = 5,6\} \cup \{(4, i) : i = 2,6\} \cup \{(5, i) : i = 2,3\} \cup \{(6, i) : i = 2,5\} \)

\( C_6 = \{(2, i) : i = 3,6\} \cup \{(3, i) : i = 2,4\} \cup \{(4, i) : i = 3,5\} \cup \{(5, i) : i = 4,6\} \cup \{(6, i) : i = 2,5\} \)

Then adjacency matrices \( M_1, M_2, M_3, M_4, M_5, \text{and} M_6 \) of \( C_1, C_2, C_3, C_4, C_5 \) and \( C_6 \) respectively are given below:
Consider the matrix:

\[
M = \begin{bmatrix}
0 & 1 & 1 & 1 & 1 \\
1 & 0 & -1 & 1 & 1 \\
1 & -1 & 0 & -1 & 1 \\
1 & 1 & 1 & 1 & -1 \\
1 & -1 & 1 & 0 & 1 \\
\end{bmatrix}
\]

Also, we see that

1. \( M_1 + M_2 + M_3 + M_4 + M_5 + M_6 = J_6 \)
2. \( M_1 + M_4 = I_6 \)
3. \( M'_1 = M_1, M'_2 = M_2, M'_3 = M_3, M'_4 = M_4, M'_5 = M_5, M'_6 = M_6 \)

We see that the following calculations:

(i) \( M_1^2 = M_1, M_2 = M_2 + 5M_1, M_3 = M_3, M_4 = M_4, M_5 = M_5, M_6 = M_6 \)
(ii) \( M_2^2 = 0, M_2M_3 = 5M_1, M_2M_4 = M_2, M_2M_5 = 2M_2, M_2M_6 = 2M_2 \)
(iii) \( M_3^2 = 0, M_3M_4 = 0, M_3M_5 = 0, M_3M_6 = 0 \)
(iv) \( M_4^2 = M_4, M_4M_5 = M_5, M_4M_6 = M_6 \)
(v) \( M_5^2 = 2M_4 + M_6, M_5M_6 = M_5 + M_6 \)
(vi) \( M_6^2 = 2M_4 + M_5 \)

Hence, product of any two adjacency matrices is some linear combinations of adjacency matrices.

Thus, the set \( C = \{C_1, C_2, C_3, C_4, C_5, C_6\} \) is a C.C. Consider the matrix \( M = 0.M_1 + 1.M_2 + 1.M_3 + 0.M_4 + 1.M_5 + (-1).M_6 \).
Another symmetric conference matrix of order 6.

\[ \begin{pmatrix}
0 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & -1 & 1 & 1 & -1 \\
1 & -1 & 0 & -1 & 1 & 1 \\
-1 & 1 & -1 & 0 & -1 & 1 \\
1 & 1 & 1 & 1 & -1 & 0 \\
1 & -1 & 1 & 1 & -1 & 0 \\
\end{pmatrix} \]

\[ \begin{pmatrix}
0 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & -1 & 1 & 1 & -1 \\
1 & -1 & 0 & -1 & 1 & 1 \\
-1 & 1 & -1 & 0 & -1 & 1 \\
1 & 1 & 1 & 1 & -1 & 0 \\
1 & -1 & 1 & 1 & -1 & 0 \\
\end{pmatrix} \]

\[ MM^T = \begin{pmatrix}
0 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & -1 & 1 & 1 & -1 \\
1 & -1 & 0 & -1 & 1 & 1 \\
-1 & 1 & -1 & 0 & -1 & 1 \\
1 & 1 & 1 & 1 & -1 & 0 \\
1 & -1 & 1 & 1 & -1 & 0 \\
\end{pmatrix} \]

\[ = 5I_6 = (6 - 1)I_6 \]

\[ MM^T = \begin{pmatrix}
0 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & -1 & 1 & 1 & -1 \\
1 & -1 & 0 & -1 & 1 & 1 \\
-1 & 1 & -1 & 0 & -1 & 1 \\
1 & 1 & 1 & 1 & -1 & 0 \\
1 & -1 & 1 & 1 & -1 & 0 \\
\end{pmatrix} \]

\[ = 5I_6 = (6 - 1)I_6 \]

\[ \Rightarrow MM^T = (6 - 1)I_6 \]

\[ M^T M = \begin{pmatrix}
0 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & -1 & 1 & 1 & -1 \\
1 & -1 & 0 & -1 & 1 & 1 \\
-1 & 1 & -1 & 0 & -1 & 1 \\
1 & 1 & 1 & 1 & -1 & 0 \\
1 & -1 & 1 & 1 & -1 & 0 \\
\end{pmatrix} \]

\[ = 5I_6 = (6 - 1)I_6 \]

\[ \Rightarrow M^T M = (6 - 1)I_6 \]

Thus \( MM^T = M^T M = (6 - 1)I_6 \)

Which show that \( M \) is another symmetric conference matrix of order 6.

References


[9] N. A. Balonin and Jennifer Seberry, Conference Matrices with Two Borders and Four circulants

