L-Covering Sets and L-Covering Polynomials of Chains

K.M.Thirunavukkarasu1, A.Vethamanickam2

1Head, Department of Mathematics, SivanthiAditanar College, Pillayarpuram, Nagercoil, India.
2Associate Professor of Mathematics, Rani Anna Government College for Women, Tirunelveli-8, India.

(Contributing Author: K.M.Thirunavukkarasu)

ABSTRACT
Let P be a finite poset. For a subset A of P, the open lower cover set of A is defined as L(A) = {x ε P| x is covered by an a ε A}. The closed lower cover set of A is defined as L[A] = L(A) ∪ A and A is called an L – covering set of P if L[A] = P. The L – covering number L(P) is the minimum cardinality of an L-covering set. Let L_n be the family of all L-covering sets of a chain P_n with cardinality i. Similarly we define U-covering and N-covering sets of P_n with cardinality i. \(\ell(P_n,i) = |L_n(i)|\), \(\alpha(P_n,i) = |U_n(i)|\), \(\nu(P_n,i) = |N_n(i)|\). In this paper, we construct \(L_n\) and obtain a recursive formula for \(\ell(P_n,i)\). Using this recursive formula we construct the polynomial \(L(P_n,x) = \sum_{i=1}^{n} \ell(P_n,i)x^i\) called L-covering polynomial of P_n.

Keywords: Poset, L-Covering set, L-Covering Polynomial.

1. INTRODUCTION
A poset P is finite if it has finite number of elements. Let P be a finite poset. The open lower cover set of A is the set L(A) = \{x ε P | x is covered by an a ε A\}. The closed lower cover set of A is the set L[A] = L(A) ∪ A. We denote L({x}) as L(x). A set \(A \subseteq P\) is a L-covering set of P if \(L[A] = P\). The L-covering number L(P) is the minimum cardinality of an L-covering set of P. A poset P is a chain if every pair of elements is comparable. Let P_n be the n element chain \(x_1 < x_2 < \ldots < x_n\). Let \(L_n\) be the family of L-covering sets of P_n with cardinality i and let \(L_n(i) = |L_n(i)|\). The polynomial \(L(P_n,x) = \sum_{i=1}^{n} \ell(P_n,i)x^i\) is called the L-covering polynomial of P_n.

2. L-covering sets of chains
In this section we construct the family of L-covering sets of chains by a recursive method. We use \([x]_i\) for the smallest integer greater than or equal to x. Let \(L_n\) be the family of L-covering sets of P_n with cardinality i. The following lemma follows from observation.

Lemma 2.1
\[L_n(i) = [2^i]_i\]
By the definition of L-covering set and by Lemma 2.1, we have the following lemma

Lemma 2.2
\(L_1 = \emptyset\) if and only if i > j or i \(\leq [\frac{j}{2}]\).

A chain connecting a and b where a < b is a simple chain if every element other than a and b in the chain has exactly one upper cover and lower cover.

The following lemma follows from observation.

Lemma 2.3
If P contains a simple chain of length 2k-1, then every L-covering set of P must contain at least k elements of the chain.

To find a L-covering set of P_n with cardinality i, we do not need to consider L-covering sets of P_n with cardinality i-1. We show this in Lemma 2.4. So, we only need to consider \(L_{n-1}\) and \(L_{n-2}\).

Lemma 2.4
If \(D \in L_{n-1}\) and if there exist \(x \in P_n\) such that \(D \cup \{x\} \notin L_n\) then \(D \in L_{n-2}\).

Proof:
Suppose that \(D \notin L_{n-2}\). Since \(D \in L_{n-1}\), if \(x \in D\) then \(D \in L_{n-2}\), a contradiction. Hence \(x \notin D\). Therefore, \(D \cup \{x\} \notin L_n\) for any \(x \in P_n\), a contradiction.

Lemma 2.5
(i) If \(L_{n-1} = L_{n-3} = \emptyset\) then \(L_{n-2} = \emptyset\).
(ii) If \(L_{n-1} \neq \emptyset\) and \(L_{n-3} \neq \emptyset\) then \(L_{n-2} \neq \emptyset\).
(iii) If \(L_{n-1} = L_{n-2} = \emptyset\) then \(L_{n-3} = \emptyset\).

Proof:
(i) Since \(L_{n-1} = L_{n-3} = \emptyset\) by Lemma 2.2, i - 1 > n-1 or i - 1 < \(\min(\frac{n-3}{2})\).
(ii) Suppose that \(L_{n-2} = \emptyset\), then by Lemma 2.2, i - 1 > n-2 or i - 1 < \(\frac{n-2}{2}\) and hence \(L_{n-3} = \emptyset\).
(iii) If i - 1 > n-2 then i - 1 > n-3 and hence \(L_{n-4} = \emptyset\), a contradiction.
Therefore, \( i-1 \leq \frac{\lfloor \frac{(n-1)}{2} \rfloor}{\lfloor \frac{(n-1)}{2} \rfloor} \) and hence \( L_{i-1}^{n-3} = \emptyset \), a contradiction.

(iii) Suppose that \( L_{i}^{n} \neq \emptyset \). Let \( D \in L_{i}^{n} \). Then \( x_{n} \in D \).

By lemma 2.3, at least one of \( x_{n+1} \) or \( x_{n+2} \) is in \( D \). If \( x_{n+1} \in D \), then \( D - \{ x_{n} \} \subseteq L_{i-1}^{n-1} \). If \( x_{n+2} \in D \), again \( D - \{ x_{n} \} \subseteq L_{i-1}^{n-2} \), a contradiction.

Lemma 2.6

If \( L_{i}^{n} \neq \emptyset \), then

(i) \( L_{i-1}^{n-1} = \emptyset \) or \( L_{i-1}^{n-2} = \emptyset \) if and only if \( n = 2k + 1 \) for some \( k \in \mathbb{N} \).

(ii) \( L_{i-2}^{n-1} = \emptyset \) and \( L_{i-1}^{n-2} = \emptyset \) if and only if \( i = n \).

(iii) \( L_{i-3}^{n-1} = \emptyset \) and \( L_{i-2}^{n-2} = \emptyset \) if and only if \( \lfloor \frac{n-1}{2} \rfloor + 1 \leq i \leq n - 2 \).

Proof:

(i) \( \Rightarrow \) since \( L_{i-1}^{n-1} \neq \emptyset \), by lemma 2.2, \( i > n-1 \) or \( i > \frac{n-1}{2} \). If \( i > n-1 \), then \( i > n \) and hence by lemma 2.2 \( L_{i}^{n} = \emptyset \), a contradiction.

Therefore, \( i > \frac{n-1}{2} \) and since \( L_{i}^{n} \neq \emptyset \)

\( i \geq \frac{n-1}{2} \).

This gives us \( n = 2k \) and \( i = k \) for some \( k \in \mathbb{N} \).

\( \Leftarrow \) If \( n = 2k \) and \( i = k \) for some \( k \in \mathbb{N} \), then by lemma 2.2.2 \( L_{i-1}^{n-1} = \emptyset \) and \( L_{i-2}^{n-2} = \emptyset \).

(ii) \( \Rightarrow \) since \( L_{i-2}^{n-1} = \emptyset \), by lemma 2.2, \( i > n-2 \) or \( i > \frac{n-2}{2} \). If \( i > \frac{n-2}{2} \), then \( i > \frac{n-1}{2} \) and hence \( L_{i-1}^{n-1} = \emptyset \), a contradiction.

Therefore, \( i > n-2 \) and so \( i = n-1 \). Also, since \( L_{i}^{n} \neq \emptyset \), \( i = n \) and hence \( i = n \).

\( \Leftarrow \) If \( i = n \), then by lemma 2.2, \( L_{i-1}^{n-1} = \emptyset \) and \( L_{i-2}^{n-2} = \emptyset \).

(iii) \( \Rightarrow \) since \( L_{i-1}^{n-1} \neq \emptyset \) and \( L_{i-2}^{n-2} = \emptyset \),

\( \frac{n-1}{2} \leq i \leq n - 2 \) and hence \( \frac{n-1}{2} \leq i \leq n - 1 \).

\( \Leftarrow \) If \( i \geq n-1 \), \( i = n \), and \( L_{i}^{n} \neq \emptyset \), and \( L_{i-1}^{n-2} \neq \emptyset \).

Theorem 2.7

For every \( n \geq 3 \) and \( i \geq \frac{n}{2} \)

(i) \( L_{i}^{n-1} = \emptyset \) and \( L_{i-1}^{n-2} = \emptyset \) if and only if \( L_{i}^{n} = \{ x_{1}, x_{2}, x_{3}, \ldots, x_{n} \} \).

(ii) \( L_{i}^{n-1} = \emptyset \) and \( L_{i-1}^{n-2} = \emptyset \) if and only if \( L_{i}^{n} = \{ x_{1}, x_{2}, x_{3}, \ldots, x_{n} \} \).

(iii) \( L_{i}^{n-1} \neq \emptyset \) and \( L_{i-1}^{n-2} = \emptyset \) if and only if \( L_{i}^{n} = \{ x_{1}, x_{2}, x_{3}, \ldots, x_{n} \} \).

Proof:

(i) \( L_{i}^{n} = \emptyset \) and \( L_{i-1}^{n-2} = \emptyset \). So, by lemma 2.6 (i), \( n = 2k \) and \( i = k \) for some \( k \in \mathbb{N} \).

Therefore, \( L_{i}^{n} = \emptyset \).

(ii) \( L_{i}^{n-1} \neq \emptyset \) and \( L_{i-1}^{n-2} = \emptyset \). So, by lemma 2.6 (ii), \( i = n \).

Therefore, \( L_{i}^{n} = \{ x_{1}, x_{2}, x_{3}, \ldots, x_{n} \} \).

(iii) \( L_{i}^{n-1} \neq \emptyset \) and \( L_{i-1}^{n-2} \neq \emptyset \). Let \( X_{i} \subseteq L_{i}^{n-1} \). Then \( X_{i} \cup \{ x_{n} \} \subseteq L_{i}^{n} \).

Therefore, \( L_{i}^{n} \subseteq \{ x_{1}, x_{2}, x_{3}, \ldots, x_{n} \} \).

Conversely, let \( X \subseteq L_{i}^{n} \). Then \( X \subseteq Y \).

By lemma 2.3, at least one of \( x_{n+1} \) or \( x_{n+2} \) is in \( Y \). If \( x_{n+1} \in Y \), then \( Y = X \cup \{ x_{n} \} \) for some \( X \subseteq L_{i}^{n-1} \). If \( x_{n+2} \in Y \), then \( Y = X \cup \{ x_{n} \} \) for some \( X \subseteq L_{i}^{n-2} \).

Therefore, \( L_{i}^{n-1} \subseteq \{ x_{1}, x_{2}, x_{3}, \ldots, x_{n} \} \).

From (1) and (2), we get (iii).

Table 1. \#(P_{n},i) the number of L-Covering sets of \( P_{n} \) with cardinality \( i \).

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3. L-covering polynomial of a chain

Let \( L(P_{n},x) = \sum_{i=1}^{n} t(P,i) x^{i} \) be the L-covering polynomial of a chain \( P_{n} \). In this section we study this polynomial.

Theorem 3.1

(i) If \( L_{i}^{n} \) is the family of L-covering sets with cardinality \( i \) of \( P_{n} \), then \( L_{i}^{n} = \{ i \} \) or \( L_{i}^{n} = \{ i-1 \} \).

(ii) For every \( n \geq 3 \), \( L(P_{n},x) = x \cdot L(P_{n-1},x) + L(P_{n-2},x) \) with initial values \( L(P_{1},x) = x \) and \( L(P_{2},x) = x^{2} + x \).

Proof:

(i) It follows from Theorem 2.7

(ii) It follows from part (i) and the definition of the L-covering Polynomial.
REFERENCES