On \(\pi\beta\)-Generalized Closed Sets in Topological Spaces

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Abstract: This paper is devoted to the study of \(\pi\beta\)-generalized closed sets and \(\pi\beta\)-generalized open sets in topological spaces and its properties.

Keywords: \(\beta\) – open set, \(\pi\beta\ g\) – closed sets, \(\pi\beta\ g\) – open sets

1. Introduction


In this paper we study the properties of generalized \(\pi\beta\)-closed sets (briefly \(\pi\beta\ g\)-closed sets).

Moreover in this paper, we defined \(\pi\beta\ g\)-open sets and obtained some of its properties.

2. Preliminaries

Definition 2.1: A subset \(A\) of a topological space \((X, \tau)\) is said to be

(a) a pre open set if \(A \subseteq \text{int}(\text{cl}(A))\) and a preclosed set if \(\text{cl}(\text{int}(A)) \subseteq A\). [16]

(b) a semiopen set if \(A \subseteq \text{cl}(\text{int}(A))\) and a semi closed set if \(\text{int}(\text{cl}(A)) \subseteq A\). [8]

(c) a \(\alpha\)-open set if \(A \subseteq \text{int}(\text{cl}(\text{int}(A)))\) and a \(\alpha\) -closed set if \(\text{cl}(\text{int}(\text{cl}(A))) \subseteq A\). [14]

(d) a semi-preopen set if \(A \subseteq \text{cl}(\text{int}(\text{cl}(A)))\) and a semi-preclosed set if \(\text{int}(\text{cl}(\text{cl}(A))) \subseteq A\). [1]

(e) a regular open set if \(A = \text{int}(\text{cl}(A))\) and a regular closed set if \(A = \text{cl}(\text{int}(A))\). [17]

(f) a generalized closed set (briefly, g-closed) if \(\text{cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is open in \(X\). [7]

(g) a semi-generalized closed set (briefly, sg-closed)

if \(\text{cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is semiopen in \(X\). [4]

(h) a generalized semi closed set (briefly, gs-closed)

if \(\text{cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is open in \(X\). [2]

(i) a generalized \(\alpha\)-closed set (briefly, g\(\alpha\)-closed) if \(\alpha\text{cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is \(\alpha\) open in \(X\). [10]

(j) a \(\alpha\) -generalized closed set (briefly, ag-closed) if \(\alpha\text{cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is open in \(X\). [9]

(k) a generalized semi -preclosed set (briefly, gsp closed) if \(\text{spcl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is semiopen in \(X\). [5]

(l) a regular generalized closed set (briefly, rg-closed)

if \(\text{cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is regular open in \(X\). [16]

(m) a generalized preclosed set (briefly, gp-closed) if \(\text{pcl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is open in \(X\). [11]

(n) a generalized preregular closed set (briefly, gpr-closed) if \(\text{pcl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is regular open in \(X\). [6]

(o) a weakly closed set (briefly, w-closed) if \(\text{cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is semiopen in \(X\). [17]

(p) a weakly generalized closed set (briefly, wg-closed) if \(\text{cl}(\text{int}(A)) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is open in \(X\). [13]

(q) a semi weakly generalized closed set (briefly, swg-closed) if \(\text{cl}(\text{int}(A)) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is semiopen in \(X\).

(r) a regular weakly generalized closed set (briefly, rwg-closed) if \(\text{cl}(\text{int}(A)) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is regular open in \(X\).

Remark 2.2: The complements of the closed sets are known as the corresponding open sets and vice versa.
Definition 2.3.[18]:
A subset A of a space (X, τ) is called:
(i) regular open if A=INT(CL(A)).
(ii) π open if A is the union of regular open sets.

3. On πβ-generalized closed sets

In this section we introduced the concept of πβ-generalized closed set in topological spaces

Definition 3.1 A subset A of a topological space (X, τ) is called πβ g-closed set (πβ -generalized closed set) if cl(int(cl(A))) ⊆ U whenever A ⊆ U and U is π open in X.

Theorem 3.2 The union of two πβ g-closed subsets of X is also πβ g-closed subset of X.

Proof: Assume that A and B are πβ g-closed set in (X, τ). Let U is π open in X such that A ∪ B ⊆ U. Then A ⊆ U and B ⊆ U. Since A and B are πβ g-closed, cl(int(cl(A))) ⊆ U and cl(int(cl(B))) ⊆ U. Hence cl(int(cl(A)) ∪ cl(int(cl(B))) ⊆ U. Therefore cl(int(cl(A ∪ B))) ⊆ U. Hence A ∪ B is πβ g-closed set in X.

Remark 3.3 The intersection of two πβ g-closed sets in (X, τ) is generally not πβ g-closed sets in X.

Example 3.4 Let X = {a, b, c} with the topology τ = {X, φ, {a}, {b}, {a, b}}. If A = {a, b} and B = {a, c}. Then A and B are πβ g-closed sets in X, but A ∩ B = {a} is not a πβ g-closed set in X.

Theorem 3.5 If a subset A of X is πβ g-closed set in X then clint(cl(A)) - A does not contain any non-empty open set in X.

Proof: Suppose that A is πβ g-closed set in X. We prove the result by contradiction. Let U be open set such that clint(cl(A)) - A ⊆ U and U ≠ φ. Now U ⊆ clint(cl(A)). Therefore, U ⊆ X-U. Since U is open in X, U is also π open in X. Since A is πβ g-closed sets in X, by definition we have clint(cl(A)) ⊆ X-U. So U ⊆ X-clint(A). Also U ⊆ clint(A). Therefore U ⊆ clint(A)(X-clint(A))= φ. This shows that U= φ which is contradiction. Hence clint(cl(A)) - A does not contains any non-empty open set in X.

Remark 3.6 The converse of the above theorem need not be true as seen from the following example.

Example 3.7 If clint(cl(A)-A contains no non-empty open set in X, then A need not be πβ g-closed. Consider X = {a, b, c} with the topology τ = {X, φ, {a}, {b}, {a, b}} and A = {a, b}. Then clint(cl(A)) - A = X - {a, b} = {c} does not contain any non-empty open set, but A is not an πβ g-closed set in X.

Theorem 3.8 If A is regular closed in (X, τ) then A is πβ g-closed subset of (X, τ).

Proof: Suppose that A ⊆ U and U is π open in X. Now U ⊆ X is open if and only if U is the union of a semi open set and pre open set. Let A be a regular closed set of (X, τ). So A = clint(cl(A)). Every regular closed set is semi open set and every semi open set is open set. Hence clint(cl(A)) ⊆ U where U is π open in X. Therefore A is πβ g-closed set in X.

Remark 3.9 The converse of the above theorem need not be true as seen from the following example.

Example 3.10 Consider X = {a, b, c} with the topology τ = {X, φ, {a}, {b}, {a, b}, {b, c}}. Let A = {a, c}. Clearly A is πβ g-closed set but not regular closed. Since, A ≠ rcl(A). This implies that A is not regular closed.

Theorem 3.11 For an element x ∈ X, the set X-{x} is πβ g-closed or open.

Proof: Suppose X-{x} is not open. Then X is the only open set containing X-{x}. This implies clint(X-{x}) ⊆ X. Hence X-{x} is an πβ g-closed set in X.

Theorem 3.12 If A is regular open and πβ g-closed, then A is regular closed and hence clopen.

Proof: Suppose A is regular open and πβ g-closed.

As every regular open set is open and Ac A, we have clint(cl(A)) ⊆ A. Since clint(cl(A)) ⊆ clint(A), we have clint(cl(A)) ⊆ clint(A). Also A ⊆ clint(A). Therefore clint(cl(A)) = A that means A is closed. Since A is regular open, A is open. Now cl(int(A)) = cl(A) = A. Therefore, A is regular closed and clopen.

Theorem 3.13 If A is regular open and rg-closed, then A is πβ g-closed set in X.

Proof: Let A be regular open and rg-closed in X. We prove that A is an πβ g-closed set in X. Let U be a π open set in such that A ⊆ U. Since A is regular open and rg-closed, we have cl(A) ⊆ A. Then cl(A) ⊆ Ac A. Hence A is πβ g-closed set in X.
Theorem 3.14 If $A$ is an $\pi\hat{\beta}$ g-closed subset in $X$ such that $A \subset B \subset \text{cl}(A)$, then $B$ is an $\pi\hat{\beta}$ g-closed set in $X$.

Proof: Let $A$ be an $\pi\hat{\beta}$ g-closed set in $X$ such that $A \subset B \subset \text{cl}(A)$. Let $U$ be a $\pi$ open set of $X$ such that $B \subset U$. Since $A$ is $\pi\hat{\beta}$ g-closed, we have $\text{cl}(A) \subset U$. Now $\text{cl}(B) \subset \text{cl}(\text{cl}(A)) = \text{cl}(A) \subset U$. Therefore $B$ is an $\pi\hat{\beta}$ g-closed set in $X$.

Remark 3.15 The converse of the above theorem need not be true as seen from the following example.

Example 3.16 Consider the topological space $(X, \tau)$, where $X = \{a, b, c\}$ be with topology $\tau = \{\emptyset, \{b\}, \{b, c\}\}$. Let $A = \{a\}$ and $B = \{a, c\}$. Then $A$ and $B$ are $\pi\hat{\beta}$ g-closed set in $(X, \tau)$, but $A \subset B$ is not in $\text{cl}(A)$.

Theorem 3.17 Let $A$ be $\pi\hat{\beta}$ g-closed in $(X, \tau)$. Then $A$ is closed if and only if $\text{cl}(A)-A$ is open.

Proof: Suppose $A$ is closed in $X$. Then $\text{cl}(A)=A$ and so $\text{cl}(A)-A=\emptyset$, which is open in $X$. Conversely, suppose $\text{cl}(A)-A$ is open in $X$. Since $A$ is $\pi\hat{\beta}$ g-closed, by Theorem 3.5, $\text{cl}(A)-A$ does not contain any non-empty open set in $X$. Then $\text{cl}(A)-A=\emptyset$, hence $A$ is closed in $X$.

Theorem 3.18 If $A$ is both open and g-closed in $X$ then it is $\pi\hat{\beta}$ g-closed set in $X$.

Proof: Let $A$ be an open and g-closed in $X$. Let $A \subset U$ and let $U$ be $\pi$ open in $X$. Now $A \subset A$. By hypothesis $\text{cl}(A) \subset A$. That is $\text{cl}(A) \subset U$. Thus $A$ is $\pi\hat{\beta}$ g-closed set in $X$.

Theorem 3.19 Every $\alpha\hat{g}$ -closed set in a topological space $X$ is $\pi\hat{\beta}$ g-closed set.

Proof: Let $A$ be a $\alpha\hat{g}$ -closed set in $(X, \tau)$ and $A \subset U$ where $\alpha$ is open. Now $\alpha$ is open implies that $U$ is $\pi$ open. Also $\text{cl}(\text{int}(A)) \subset \text{cl}(A) \subset \alpha \text{cl}(A) \subset U$. Hence $A$ is $\pi\hat{\beta}$ g-closed set in $X$.

Remark 3.20 The converse of the above theorem need not be true as seen from the following example.

Example 3.21 Consider the topological space $(X, \tau)$, where $X = \{a, b, c\}$ be with topology $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. Then let $A = \{a\}$ is $\pi\hat{\beta}$ g-closed set in $(X, \tau)$, but not $\alpha\hat{g}$ -closed set in $X$.

4. On $\pi\hat{\beta}$ -generalized open sets and $\pi\hat{\beta}$ -generalized neighbourhoods

In this section, we introduce an study $\pi\hat{\beta}$ g – open sets in topological spaces and obtain some of their properties. Also, we introduce $\pi\hat{\beta}$ g– neighborhood (briefly $\pi\hat{\beta}$ g – nbhd) in topological spaces by using the notion of $\pi\hat{\beta}$ g – open sets.

Definition 4.1 A subset $A$ in $X$ is called $\pi\hat{\beta}$ generalized open (briefly $\pi\hat{\beta}$ g – open) in $X$ if $A^c$ is $\pi\hat{\beta}$ g – closed in $X$. We denote the family of all $\pi\beta$ g – open sets in $X$ by $\pi\hat{\beta}$ gO$(X)$.

Theorem 4.2 If $A$ and $B$ are $\pi\hat{\beta}$ g – open sets in a topological space $X$. Then $A \cap B$ is also $\pi\hat{\beta}$ g – open set in $X$.

Proof: Let $A$ and $B$ be $\pi\hat{\beta}$ g – open sets in a space $X$. Then $A^c$ and $B^c$ are $\pi\hat{\beta}$ g – closed set in $X$. By Theorem 3.2, $A^c \cup B^c = (A \cap B)^c$ is also $\pi\hat{\beta}$ g – closed set in $X$. Therefore $A \cap B$ is also $\pi\hat{\beta}$ g – open set in $X$.

Definition 4.3 Let $X$ be a topological space and let $x \in X$. A subset $N$ of $X$ is said to be a $\pi\hat{\beta}$ g – nbhd of $x$ iff there exists a $\beta\hat{\beta}$ g-open set $G$ such that $x \in G \subset N$.

Definition 4.4 A subset $N$ of space $X$, is called a $\pi\hat{\beta}$ g – nbhd of $\text{cl}(A)$ if there exists a $\pi\hat{\beta}$ g-open set $G$ such that $A \subset G \subset N$.

Remark 4.5 The $\pi\hat{\beta}$ g – nbhd $N$ of $x \in X$ need not be a $\pi\hat{\beta}$ g-open set in $X$.

Example 4.6 Consider the topological space $(X, \tau)$, where $X = \{a, b, c\}$ be with topology $\tau = \{X, \emptyset, \{a\}, \{b\}, \{c\}, \{a, c\}\}$.

The $\pi\hat{\beta}$ gO$(X) = \{X, \emptyset, \{a\}, \{b\}, \{c\}, \{a, c\}\}$. Note that $\{a, b\}$ is not a $\pi\hat{\beta}$ g – open set in $(X, \tau)$, but it is a $\pi\hat{\beta}$ g – nbhd of $\{a\}$. Since $\{a\}$ is a $\pi\hat{\beta}$ g-open set such that $a \in \{a\} \subset \{a, b\}$.

Theorem 4.7 Every nbhd $N$ of $x \in X$ is a $\pi\hat{\beta}$ g –
nbhd of X.

Proof: Let N be a nbhd of point x∈X. To prove that N is a π_f g – nbhd of x. By definition of nbhd, there exists an open set G such that x∈G⊂N. As every open set is π_f g – open set G such that x∈G⊂N. Hence N is π_f g – nbhd of X.

Remark 4.8 In general, a π_f g – nbhd N of x∈X need not be a nbhd of x in X, as seen from the following example.

Example 4.9 Consider the topological space (X,τ), where X = {a, b, c} be with the topology τ = {X, φ, {c}}. The π_f gO(X) = {X, φ, {a}, {b}, {c}, {b, c}, {a, c}}. The set {a, b} is π_f g – nbhd of the point b, since the π_f g – open set {b} is such that b ∈ {b} ⊂ {a, b}. However the set {a, b} is not a nbhd of the point b, since no open set G exists such that b ∈ G ⊂ {a, b}.

Theorem 4.10 If a subset N of a space X is π_f g – open, then N is a π_f g – nbhd of each of its points.

Proof: Suppose N is π_f g – open. Let x∈N. We claim that N is π_f g – nbhd of x. For N is a π_f g – open set such that x∈N⊂N. Since x is an arbitrary point of N, it follows that N is a π_f g – nbhd of each of its points.

Remark 4.11 The converse of the above theorem need not be true as seen from the following example.

Example 4.12 Consider the topological space (X,τ), where X = {a, b, c} be with the topology τ = {X, φ, {c}}. Then π_f gO(X) = {X, φ, {a}, {b}, {c}, {b, c}, {a, c}}. The set {a, b} is π_f g – nbhd of the point a, since the π_f g – open set {a} is such that a ∈ {a} ⊂ {a, b}. Also the set {a, b} is π_f g – nbhd of the point b, since the π_f g – open set {b} is such that b ∈ {b} ⊂ {a, b}. That is, {a, b} is π_f g – nbhd of each of its points. However the set {a, b} is not a π_f g – open set in X.

Theorem 4.13 Let X be a topological space. If F is a π_f g – closed subset of X and x∈F. Prove that there exists a π_f g – nbhd N of x such that N∩F=φ.

Proof: Let F is a π_f g – closed subset of X and x∈F. Then F is π_f g – open set of X. Therefore, F contains a π_f g – nbhd of each of its points. Hence there exists a π_f g – nbhd N of x such that N⊂F. That is N∩F=φ.

References