On $\delta$ - lower semi precontinuous functions

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Abstract: In this paper, the concept of $\delta$ - lower semi precontinuous functions is to be introduced. Some characterization theorems and their basic properties are also to be investigated. It is note that $\delta$ - lower semi precontinuous functions play an important role in defining $\delta$ - preinduced fuzzy supra topological spaces. The connection between the $\delta$ - preinduced fuzzy supra topology and its corresponding topological space is to be studied. The relationship between $\delta$ - preinduced fuzzy supra topological spaces and that of the induced fuzzy topological spaces due to Lowen are to be investigated. Lastly some applications are to be shown.

Key Words: $\delta$ - lower semi precontinuous function, induced fuzzy topological spaces, $\delta$ - induced fuzzy topological spaces, $\delta$ - preinduced fuzzy topological spaces.

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1. Introduction

The concept of induced fuzzy topological space (IFTS) was introduced by Weiss [16]. Lowen [8] called these spaces as topologically generated space. The concept of fuzzy supra topological space was introduced by Abd El-Monsef and Ramadan [1]. The notion of lower semi continuous functions [16] plays an important role in defining the above concept. The concept of lower semi precontinuous functions was introduced by Mukherjee [11]. Mukherjee and Halder [12] introduced the concepts of $\delta$ - lower semi continuous functions and $\delta$ - induced fuzzy topological spaces. The aim of this chapter is to introduce a new class of functions, called $\delta$ - lower semi precontinuous functions as a generalization of $\delta$ - precontinuous functions. Also the purpose of this chapter is to introduce the notion of $\delta$ - preinduced fuzzy supra topological spaces. The notion of $\delta$ - lower semi precontinuous functions plays an important role in defining $\delta$ - preinduced fuzzy supra topological spaces.

In section 3, I would introduce the concept of $\delta$ - lower semi precontinuous functions. Some of their characterization theorems and basic properties are also to be studied. In section 4, $\delta$ - preinduced fuzzy supra topological spaces is to be defined. I would also study some connections between the properties of a topological space $(X, T)$ and that of the corresponding $\delta$ - preinduced fuzzy supra topological space $(X, D(T))$. In section 5, I introduce and study $\delta$ - p - initial spaces.

2. Preliminaries

In this section I would like to mention some known results and definitions for ready references.

Definition 2.1. [15] A set $A$ of a topological space $(X, T)$ is called regular open if $A = \text{intcl}(A)$.

Definition 2.2. [9] A set $A$ of a topological space $(X, T)$ is called preopen if $A \subseteq \text{intcl}(A)$.

Definition 2.3. [15] A set $A$ of a topological space $(X, T)$ is called $\delta$ - open if for each point $x \in A$, there exists a regular open set $W$ such that $x \in W \subseteq A$. The complement of a $\delta$ - open set is said to be $\delta$ - closed.

Definition 2.4. [5] A set $A$ of $X$ is called $\delta$ - preopen if $A \subseteq \text{intcl}(A)$. The complement of a $\delta$ - preopen set is said to be $\delta$ - preclosed.

Definition 2.5. [6] The strong $r$ - cut and weak $r$ - cut of $\alpha$ are defined by $s_r(\alpha) = \{x \in X : \alpha(x) > r\}$ and $W_r(\alpha) = \{x \in X : \alpha(x) \geq r\}$ where $\alpha \in I^X$ and $r \in I = [0, 1]$.

Definition 2.6. [16] A function $f : (X, T) \rightarrow (R, U)$ is said to be lower semi continuous (LSC) at a point $x_0$ of $X$ if and only if for every $\varepsilon > 0$ there exists an open neighbourhood $N(x_0)$ such that for every $x \in N(x_0)$ implies $f(x) > f(x_0) - \varepsilon$.

Definition 2.7. [16] Let $(X, T)$ be a topological space. The collection $W(T)$ of all lower semi continuous (LSC) functions $f : (X, T) \rightarrow I$ (I is the unit closed interval) forms a fuzzy topology on $X$. Then $(X, W(T))$ is known as an induced fuzzy topological (IFT) space.

Definition 2.8. [11] A function $f : (X, T) \rightarrow (R, U)$ is said to be lower semi precontinuous (LSPC) at a point $x_0$ of $X$ if and only if for every $\varepsilon > 0$ there exists an preopen neighbourhood $N(x_0)$ such that for every $x \in N(x_0)$ implies $f(x) > f(x_0) - \varepsilon$.

Definition 2.9. [12] A function $f : (X, T) \rightarrow (R, U)$ is said to be $\delta$ - lower semi continuous at a point $x_0$ of $X$ if and only if for every $\varepsilon > 0$, there exists a $\delta$ - open neighbourhood $N(x_0)$ such that for every $x \in N(x_0)$ implies $f(x) > f(x_0) - \varepsilon$.

Definition 2.10. [12] Let $(X, T)$ be a topological space. The collection $W(T)$ of all $\delta$ - lower semi continuous (LSC) functions $f : (X, T) \rightarrow I$ (I is the unit closed interval) forms a fuzzy topology on $X$. Then $(X, W(T))$ is known as an $\delta$ - induced fuzzy topological space.
3. $\delta$ - lower (upper) semi precontinuous functions

In this section, I would introduce a new class of functions as a generalization of $\delta$ - precontinuous functions. A function $f : (X, T_1) \to (Y, T_2)$ from a topological space $(X, T_1)$ to another topological space $(Y, T_2)$ is called $\delta$ - precontinuous if and only if the inverse image of every open subset in $Y$ is $\delta$ - preopen in $X$. By $X$, we mean a topological space $(X, T)$.

Definition 3.1. A function $f : X \to R$, where $R$ is the real line is said to be $\delta$ - lower semi precontinuous (LSPC) at $x_0$ of $X$ if and only if for each $\varepsilon > 0$, there exists a $\delta$ - preopen neighbourhood $N(x_0)$ such that for every $x \in N(x_0)$ implies $f(x) > f(x_0) - \varepsilon$.

A function $f : X \to R$ is said to be $\delta$LSPC if it is so at each point of $X$.

Definition 3.2. A function $f : X \to R$, where $R$ is the real line is said to be $\delta$ - upper semi precontinuous (USPC) at $x_0$ of $X$ if and only if for each $\varepsilon > 0$, there exists a $\delta$ - preopen neighbourhood $N(x_0)$ such that for every $x \in N(x_0)$ implies $f(x) < f(x_0) + \varepsilon$.

A function $f : X \to R$ is said to be $\delta$USPC if it is so at each point of $X$.

Theorem 3.3. The necessary and sufficient condition for a real valued function $f$ to be $\delta$LSPC is that for all $a \in R$, the set \{x \in X : f(x) > a\} is $\delta$ - preopen.

Proof: Let $f$ be $\delta$LSPC function and let $x_0 \in X$ and $N(x_0)$ be the $\delta$ - preopen nbhd. of a point $x_0$ of $X$. Then $f(x_0)$ is a real number so that $f(x_0) - \varepsilon$ is a fixed real number in $R$ for a point $x_0$ of $X$. By definition $f(x) > f(x_0) - \varepsilon = a$ (say). Hence the set of all points $x$ for which $f(x) > a$ is $\delta$ - preopen.

Conversely, let $x \in X$ be such that $f(x) > a$ is $\delta$ - preopen and let $x_0$ be any point in $X$. Let us choose a real number $\varepsilon$ such that $f(x_0) - \varepsilon = a = a \in R$. From the given condition the set \{x \in X : f(x) > f(x_0) - \varepsilon\} is $\delta$ - preopen which implies that if $\{x \in A \subseteq X\}$ is $\delta$ - preopen, then the condition $f(x) > f(x_0) - \varepsilon$ is true which shows that $f$ is $\delta$LSPC.

Theorem 3.4. The necessary and sufficient condition for a real valued function $f$ to be $\delta$USPC is that for all $a \in R$, the set \{x \in X : f(x) \leq a\} is $\delta$ - preclosed.

Proof: From Theorem 3.3, it follows that the necessary and sufficient condition for a real valued function $f$ to be $\delta$LSPC is that for all $a \in R$, the set \{x \in X : f(x) \leq a\} is $\delta$ - preclosed, being the complement of $\delta$ - preopen. Hence the theorem.

Theorem 3.5. A function $f$ from a topological space $(X, T)$ onto a space $(R, \sigma_r)$, where $\sigma_r = \{[r, \infty) : r \in R\}$ is $\delta$LSPC if and only if the inverse image of any open subset of $(R, \sigma_r)$ is $\delta$ - preopen in $(X, T)$.

Proof: Let $f : (X, T) \to (R, \sigma_r)$ be a $\delta$LSPC function and $N(x_0)$ be any $\delta$ - preopen nbhd. of a point $x_0$ of $X$. Then for any point $x \in N(x_0)$, $f(x) > f(x_0) - \varepsilon$. It is clear that the set \{x \in X : f(x) > f(x_0) - \varepsilon\} = N(x_0), which is $\delta$ - preopen. Taking $f(x_0) - \varepsilon = r$, we have $\{x \in X : f(x) > r\}$, which is same as the set \{x \in $f^{-1}(r, \infty) : f(x) > r\}$. Thus for any open set $(r, \infty)$ in $(R, \sigma_r)$, $f^{-1}(r, \infty) = \{x \in X : f(x) > r\}$ is $\delta$ - preopen.

Conversely, let the inverse image of an open subset $(r, \infty)$ is $\delta$ - preopen in $(X, T)$. Then $f^{-1}(r, \infty)$ consists of those points of $X$ whose images are $> r$, i.e., $f^{-1}(r, \infty) = \{x \in X : f(x) > r\}$ is $\delta$ - preopen. Thus by Theorem 3.3., $f$ is $\delta$LSPC function.

Theorem 3.6. The characteristic function of a $\delta$ - preopen set is $\delta$LSPC function.

Proof: Let $A$ be a $\delta$ - preopen set. The characteristic function of $A$ is defined by

$$\mu_A = \begin{cases} 1, & x \in A \\ 0, & x \in X - A \end{cases}$$

We have to show that $\mu_A$ is $\delta$LSPC function. We wish to show that \{x : \mu_A(x) \leq r\} is $\delta$ - preclosed, for each $r \in R$. For $r < 0$, the set \{x : \mu_A(x) \leq r\} = \emptyset, which is $\delta$ - preclosed. For $0 \leq r < 1$, the set \{x : \mu_A(x) \leq r\} = X - A, which is again $\delta$ - preclosed, being complement of $\delta$ - preopen subset of $A$. For $r \geq 1$, the set \{x : \mu_A(x) \leq r\} = X, which is $\delta$ - preclosed. Hence $\mu_A$ is $\delta$LSPC function.

Hence the theorem.

Theorem 3.7. If $\{f_j : j \in I\}$ is an arbitrary family of $\delta$LSPC functions, then the function $g$ defined by $g(x) = \text{Sup}(f_j(x))$ is $\delta$LSPC.

Proof: Let $r \in R$ and $g(x) < r$, then $f_j < r$, for all $j \in I$. Now $\{x \in X : g(x) \leq r\} = \cap \{x \in X : f_j(x) \leq r\}$. But each $f_j$ being $\delta$ - lower semi precontinuous, by Theorem 3.4., each set $\{x \in X : f_j(x) \leq r\}$ is $\delta$ - preclosed in $X$. Now we know that an arbitrary intersection of $\delta$ - preclosed sets is $\delta$ - preclosed. Therefore $g$ is $\delta$LSPC.

Theorem 3.8. If $f_1, f_2, f_3, \ldots, f_n$ are $\delta$LSPC functions, then the function $h$ defined by $h(x) = \inf(f_j(x))$ is not $\delta$LSPC, where $j = 1 \to n$.

Proof: The theorem follows from the fact that an arbitrary intersection of $\delta$ - preopen subsets may not be $\delta$ - preopen.

Result 3.9. Every $\delta$-lower semi continuous ($\delta$-LSC) function is $\delta$-lower semi precontinuous ($\delta$LSPC) function.

Proof: Since every $\delta$-open subset is $\delta$-preopen, the result follows immediately.

The converses of the above result is not true, which can be shown from the following example.

Example 3.10. Let $X = \{a, b, c\}$ and $Y = \{0, 1, 2\}$. Let $t_1 = \{X, \emptyset, \{a, b\}\}$ and $t_2 = \{Y, \emptyset, \{1\}, \{2\}, \{1, 2\}\}$ be the topologies on $X$ and $Y$ respectively. A function $f : X \to Y$ is defined by $f(a) = 1, f(b) = 2, f(c) = 0$. Now $f^{-1}(0) = \{c\}, f^{-1}(1) = \{b\}, f^{-1}(1, 2) = \{a, b\}$. We observe that $\{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}$ are preopen subsets in $(X, t_1) [11]$. Hence each of them is $\delta$-preopen set in $(X, t_1)$. We fix $r = 1$, then the set $\{x \in X : f(x) > 1\}$ is $\delta$ - preopen in $X$ (since the inverse
image of the open subset $\{2\}$ in $Y$ is $\delta$ - preopen in $X$, thus by Theorem 3.3., $f$ is $\delta$-LSCP. But $f$ is not $\delta$-LSC since $\{b\} \notin \tau_1$. 

4. $\delta$ - preinduced fuzzy supra topological spaces

The notion of induced fuzzy topology $W(T)$ was due to Weiss [16]. It is the collection of all semicontinuous functions from a topological space $(X, T)$ to the unit closed interval $I = [0, 1]$. If $A \in T$, then $I_A \in W(T)$. Abd El-Monsef and Ramadan [1], introduced the concept of fuzzy supra topology as follows:

A family $T^* \subset I^*$ is called a fuzzy supra topology on $X$, if $0, 1 \in T^*$ and $T^*$ is closed under arbitrary union.

In this section, the notion of $\delta$ - preinduced fuzzy supra topology ($\delta$PIFST) is to be introduced. Its properties and the concept of fuzzy supra $\delta$ - precontinuity in $\delta$PIFST spaces are to be studied.

Theorem 4.1. Let $(X, T)$ be a topological space. The family of all $\delta$ - lower semi precontinuous functions from the topological space $(X, T)$ to the unit closed interval $I = [0, 1]$ forms a fuzzy supra topology on $X$.

Proof : Let $D(T)$ be the collection of all $\delta$LSCP functions from $(X, T)$ to $I$. We will now prove that $D(T)$ forms a fuzzy supra topology on $X$.

(i) Since $X$ is open, it is $\delta$ - preopen and thus $I_X \in D(T)$ (by Theorem 3.6.).

(ii) $\Phi$ is $\delta$ - preopen, thus $I_\emptyset \in D(T)$.

(iii) Let $\{T_j : j \in I\}$ be an arbitrary family of $\delta$LSCP functions, then $\operatorname{Sup}(f(x)) \in D(T)$ [by Theorem 3.6.]. Thus $D(T)$ forms a fuzzy supra topology.

Definition 4.2. The fuzzy supra topology obtained as above is called $\delta$ - preinduced fuzzy supra topology ($\delta$PIFST) and the space $(X, D(T))$ is called the $\delta$ - preinduced fuzzy supra topological ($\delta$PIFST) space. The members of $D(T)$ are called fuzzy supra $\delta$-open subsets.

Example 4.3. Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a, b\}\}$ be a topology on $X$. Besides the members of $\tau$, it follows that $\{a\}, \{b\}, \{a, c\}, \{b, c\}$ are also preopen subsets, hence $\delta$ - preopen subsets. Thus $1_a, 1_X, 1_{\{a\}}, 1_{\{b\}}, 1_{\{a, b\}}, 1_{\{a, c\}}, 1_{\{b, c\}}$ are all $\delta$LSCP. Then the collection of all these functions form a $\delta$PIFST on $X$.

Lemma 4.4. If $A$ is $\delta$ - preopen in $(X, T)$, then $\mu_A \in D(T)$.

Proof : By the Theorem 3.6., the lemma follows immediately.

Lemma 4.5. If $W(T)$ is an induced fuzzy topology and $D(T)$ is a $\delta$ - preinduced fuzzy supra topology on $X$, then $W(T) \subset D(T)$.

Proof : If $\alpha \in W(T)$, i.e., $\alpha$ is lower semi continuous function. Since every lower semi continuous function is $\delta$ - lower semi precontinuous, $\alpha \in D(T)$. Hence $W(T) \subset D(T)$.

Theorem 4.6. A fuzzy subset $A$ in a $\delta$PIFST space $(X, D(T))$ is fuzzy supra open if and only if for each $r \in I$, strong $r$ - cut $\sigma_r(A) = \{x \in X : A(x) > r\}$ is $\delta$ - preopen in the topological space $(X, T)$.

Proof : A fuzzy subset $A$ is fuzzy supra open in $(X, D(T))$ if $A \in D(T)$. Now $A \in D(T)$ if and only if $A$ is $\delta$LSCP. Again $A$ is $\delta$LSCP if and only if for each $r \in I$, $\{x \in X : A(x) > r\}$ is $\delta$ - preopen in $(X, T)$, i.e., $\sigma_r(A)$ is the $\delta$ - preopen in $(X, T)$.

Corollary 4.7. A fuzzy subset $A$ in a $\delta$PIFST space $(X, D(T))$ is fuzzy supra closed if and only if for each $r \in I$, the weak $s$ - cut $W_r(A) = \{x \in X : A(x) \geq s\}$ is $\delta$ - preclosed in the topological space $(X, T)$.

Definition 4.8. A function $f : (X, D(T_1)) \to (X, D(T_2))$ is a $\delta$PIFST space to another $\delta$PIFST space is said to be fuzzy supra $\delta$ - precontinuous if $f^*(A) \in D(T_2)$ for every $A \in D(T_1)$.

Definition 4.9. A function $f : X \to Y$ from a topological space $(X, T_1)$ to another topological space $(Y, T_2)$ is said to be $\delta$ - preirresolute if and only if the inverse image of each $\delta$ - preopen subset in $Y$ is $\delta$ - preopen in $X$.

Theorem 4.10. A function $f : (X, D(T_1)) \to (X, D(T_2))$ is fuzzy supra $\delta$ - precontinuous if and only if $f : (X, T_1) \to (Y, T_2)$ is $\delta$ - preirresolute.

Proof : Let $f$ be fuzzy supra $\delta$ - precontinuous and $A$ is $\delta$ - preopen in $(Y, T_2)$. Then $f^{-1}(A) = \{x \in X : \mu_A(f(x)) = 1\} = \{x \in X : f^*(\mu_A(x)) > r, 0 < r < 1\} = \sigma_r(f^*(\mu_A)).$

Since $A$ is $\delta$ - preopen, by Lemma 4.4., $\mu_A \in D(T)$, thus, $f^*(\mu_A) \in D(T_1)$. Hence by Theorem 4.6., $\sigma_r(f^*(\mu_A))$ is $\delta$ - preopen in $(X, D(T_1))$ which implies $f$ is $\delta$ - preirresolute.

Conversely, let us consider $f : (X, D_1) \to (Y, D_2)$ to be $\delta$ - preirresolute and $B \in D(T_2)$. Now, for any $s > 0$, $\sigma_s(f^{-1}(B)) = \{x \in X : f^*(B)(x) > s, 0 < s < 1\} = \{x \in X : f^{-1}(B)(x) > s\} = f^*(B^c(s, \infty)).$

Since $B \in D(T_2)$, $B$ is $\delta$LSCP, therefore, $B^c(s, \infty)$ is $\delta$ - preopen in $(Y, T_2)$. Hence by hypothesis, $f^*(B^c(s, \infty))$ is $\delta$ - preopen in $(X, T_1)$, i.e., $\sigma_s(f^*(B))$ is $\delta$ - preopen in $(X, T_1)$, which implies $f^*(B) \in D(T_1)$.

Hence $f$ is fuzzy supra $\delta$ - precontinuous.

5. $\delta$ - $p$ - initial spaces

The notion of initial fuzzy topology was first introduced by Lowen [8] and was further studied by Chang Ming [10]. Mukherjee [11] introduced and studied the concept of $p$ - initial topology. In this section, the concepts $\delta$ - $p$ - initial topology are to be introduced in a $\delta$PIFST and some of its properties are to be investigated. The relationship between the topologies of $\delta$ - $p$ - initial space and an $p$ - initial space is also to be studied.

Definition 5.1. Let $(X, D(T))$ be a $\delta$PIFST spaces. The family $\sigma_r(A) : A \in D(T), r \in I$ of $\delta$ - preopen subsets of $X$ forms a subbase of some topology of $X$, called $\delta$-
- p-initial topology of $D(T)$ and is denoted by $i_\delta p(T')$ where $T' = D(T)$. Then $(X, i_\delta p(T'))$ is known as $\delta$ - p-initial topological space or $\delta$ - p-initial space.

**Example 5.2.** Let $X = \{a, b, c\}$ and $\tau = \{X, \phi\}, \{a, b\}$ be a topology on $X$. Here preopen sets, hence $\delta$-preopen sets of $(X, \tau)$ are $\{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. $i_\delta p(T) = \{X, \{a\}, \{b\}, \{a, b\}\}$.

**Lemma 5.3.** $i(W)(T) \subseteq i_\delta p(D(T))$, where $i_\delta p(D(T))$ is the $\delta$-p-initial topology and $i(W)$ is the initial topology on $X$.

**Proof :** Straight forward.

**Theorem 5.4.** Let $(X, T_1)$ and $(Y, T_2)$ be two topological spaces and $i_\delta p(D(T_1))$ and $i_\delta p(D(T_2))$ be the $\delta$ - p - initial topological spaces on $X$ and $Y$ respectively. Then

(a) If $f : (X, T_1) \rightarrow (Y, T_2)$ is $\delta$ - precontinuous, then $f : (X, i_\delta p(D(T_1))) \rightarrow (Y, i_\delta p(D(T_2)))$ is $\delta$ - continuous.

(b) If $f : (X, T_1) \rightarrow (Y, T_2)$ is $\delta$ - preirresolute, then $f : (X, i_\delta p(D(T_1))) \rightarrow (Y, i_\delta p(D(T_2)))$ is $\delta$ - continuous.

**Proof :** Straight forward.

**References**