Fixed Point Theorems for occasionally weakly compatible self maps in Generalized Fuzzy Metric Spaces
Sukanya K P*1, Dr.Sr.MagieJose*2

*Research Scholar, Department of Mathematics, St.Mary’s College, Thrissur, Kerala, India
**Associate Professor, Department of Mathematics, St.Mary’s College, Thrissur , Kerala, India

Abstract: In this paper we are proving common fixed Point theorems for occasionally weakly compatible self maps in Generalized fuzzy metric spaces.

Keywords —Fuzzy Metric Space, G-Metric Space, Q- Fuzzy Metric Space, weakly compatible and occasionally weakly compatible self mappings

I. INTRODUCTION

In Mathematics, the concept of Fuzzy set was introduced by L.A Zadeh[2]. It is a new way to represent vagueness in our daily life. In 1975 Kramosil and Michalek[10] introduced the concept of fuzzy metric spaces. After that several fixed point theorems have been proved in fuzzy metric spaces.


In 2006, Mustafa Z. and B. Sims[7] presented a definition of G-metric space. After that several fixed point results have been proved in G-metric spaces. Later, Guangpeng Sun and Kai Yang[5] introduced the notion of Q-fuzzy metric space, which can be considered as a generalization of fuzzy metric space. They also prove fixed point theorems in Q-fuzzy metric spaces.

In this paper we prove common fixed point theorems for four mappings under occasionally weakly compatible condition in generalized fuzzy metric space. To prove our results we use the concept of occasionally weakly compatible maps due to Al-Thagafi and NaseerShahzad[11]. While proving the results we take Q-fuzzy metric spaces which is a generalization of fuzzy metric spacesequed to Guangpeng Sun and Kai Yang[5]. The results presented in this paper generalize and improve some known results due to C.T Aage and J.N Salunke[4].

II. PRELIMINARY NOTES

Definition 2.1[2]: A fuzzy set A in X is a function with domain X and values in [0,1]

Definition 2.2[6]: A binary operation *:[0,1]×[0,1] →[0,1] is a continuous t-norm if * satisfies the following conditions

- * is commutative and associative
- * is continuous
- a*b = a for all a, b ∈ [0,1]
- a*b ≤ c*d whenever a ≤ c and b ≤ d for all a, b, c, d ∈ [0,1]

Definition 2.3[1]: A 3-tuple (X, M, ◦) is called a fuzzy metric space if X is an arbitrary set, ◦ is a continuous t-norm and M is a fuzzy set in X × [0,∞) satisfying the following conditions

- M(x,y,t) > 0
- M(x,y,t) = 1 for all t > 0 and only if x = y
- M(x,y,t) = M(y,x,t)
- M(x,y,t)*M(y,z,s) ≤ M(x,z,t+s)
- M(x,y,0) = 0 if and only if x = y.

M(x,y,t) denotes the degree of nearness between x and y with respect to t.

Definition 2.4[7]: Let X be a nonempty set and let G:X×X→R+ be a function satisfying the following

- G(x,y,z) = 0 if x = y = z
- 0 ≤ G(x,y) for all x,y ∈ X
- G(x,y,z) ≤ G(x,y) + G(y,z)
- G(x,y,z) = G(y,z,x) = G(y,x,z) = ... (symmetry in all three variables)
- G(x,y,z) ≤ G(x,a,a) + G(a,y,z) for all x, y, z, a ∈ X (Rectangle inequality)

Then the function is called a generalized metric, or more specifically a G-metric on X and the pair (X,G) is a G-metric space.

Example: Let G:X×X→R+ be a function defined by G(x,y,z) = d(x,y) + d(y,z) + d(x,z). Then (X,G) is a G-metric space.
**Definition 2.5[5]:** A 3-tuple $(X, Q, *)$ is called a $Q$-fuzzy metric space if $X$ is an arbitrary set, $*$ is a continuous t-norm and $Q$ is a fuzzy set in $X^3 \times (0, \infty)$ satisfying the following conditions for each $x, y, z, a \in X$ and $t, s > 0$

- $Q(x, y, z, t) > 0$ and $Q(x, y, z, t) \leq Q(x, y, z, t)$ for all $x, y, z \in X$ with $z \neq y$
- $Q(x, y, z, t) = 1$ for all $t > 0$ if and only if $x = y = z$

Then $(X, Q, *)$ is a fuzzy metric space.

**Example:** Let $X$ be a non-empty set and $G$ be the $G$-metric on $X$. The $t$-norm is $ab = \alpha$ for all $a, b \in [0, 1]$. For each $t > 0$

$$Q(x, y, z, t) = Q(\alpha, x, y, z, t).$$

Then $(X, Q, *)$ is a $Q$-fuzzy metric space.

**Lemma 2.6[5]:** If $(X, Q, *)$ be a $Q$-fuzzy metric space, then $Q(x, y, z, t)$ is non-decreasing with respect to $t$ for all $x, y, z \in X$.

**Definition 2.7[5]:** Let $(X, Q, *)$ be a $Q$-fuzzy metric space. A sequence $(x_n)$ in $X$ converges to a point $x$ if and only if $Q(x_m, x_n, t) \to 1$ as $n \to \infty$, $m \to \infty$, for each $t > 0$.

**Definition 2.8[5]:** Let $f$ and $g$ be self mappings on a $Q$-fuzzy metric space $(X, Q, *)$. Then the mappings are said to be compatible if $\lim_{n \to \infty} Q(gf \circ x_n, \circ x_n, t) = 1$, for all $t > 0$ whenever $(x_n)$ is a sequence in $X$ such that $\lim_{n \to \infty} f \circ x_n = \lim_{n \to \infty} g \circ x_n = z$ for some $z \in X$.

**Definition 2.9[4]:** Let $X$ be a set. Let $f$ and $g$ be self mappings on $X$. A point $x$ in $X$ is called a coincidence point of $f$ and $g$ if and only if $fx = gx$. In this case $w = fx = gx$ is called point of coincidence of $f$ and $g$.

**Definition 2.10[4]:** A pair of self mappings $(f, g)$ is said to be weakly compatible if they commute at the coincidence points, that is if $fu = gu$ for some $u \in X$, then $fgu = gfu$.

**Definition 2.11[9]:** Two self maps $f$ and $g$ of a set are occasionally weakly compatible (owc) if there is a point $x$ in $X$ which is a coincidence point of $f$ and $g$ at which $f$ and $g$ commute.

**Lemma 2.12[4]:** Let $X$ be a set, $f, g$ owc self maps of $X$. If $f$ and $g$ have a unique point of coincidence, $w = fx = gx$, then $w$ is the unique common fixed point of $f$ and $g$.

**III. THEOREMS**

**Theorem 3.1:** Let $(X, Q, *)$ be a complete $Q$-fuzzy metric space and let $A, B, S$ and $T$ be self mappings of $X$. Let the pair $(A, S)$ and $(B, T)$ be occasionally weakly compatible (owc). If there exist a $k \in (0, 1)$ such that

$$Q(Ax, By, Bz, kt) \geq \varphi((Q(Sx, Ty, Tz, t), Q(Sx, Ty, Tz, t), Q(Ax, Ty, Tz, t), Q(Ax, Ty, Tz, t)).$$

For all $x, y, z \in X$ and $\varphi: [0, 1]^2 \to [0, 1]$ such that $\varphi(t, 1, t) > t$ for $0 < t < 1$. Then there exists a unique common fixed point of $A, B, S$ and $T$.

**Proof:** The pair of self mappings $(A, S)$ and $(B, T)$ be occasionally weakly compatible (owc). So there are points $x, y, z \in X$ such that $Ax = Sx$ and $By = Ty$. We claim that $Ax = By$. If not by inequality (1) we have

$$Q(Ax, Ty, Ty, Ty, t) = \varphi((Q(Ax, By, By, t), Q(Ax, Ty, By, t), Q(Ax, By, By, t), Q(Ax, By, By, t))).$$

Therefore $Ax = Az$. And the unique point of coincidence of $A$ and $S$. Then by Lemma 2.12 we have $w$ is the common fixed point of $A$ and $S$. Similarly there is a unique point $z \in X$ such that $Bz = Tz$. Assume that $w \neq x$. We have

$$Q(w, z, z, t) = Q(Aw, Bz, Bz, t)$$

which is a contradiction. Therefore $Ax = By$, $teAx = Sx = By = Ty$.

Let there exist another point $z$ such that $Az = Sz$. Then by inequality (1) we have $z = Sz = By = Ty$. Therefore $Ax = Az$ i.e. $Ax = Sx$ is the unique point of coincidence of $A$ and $S$. Then by Lemma 2.12 $w$ is the common fixed point of $A$ and $S$. Similarly there is a unique point $z \in X$ such that $Bz = Tz$. Assume that $w \neq z$. We have

$$Q(w, z, z, t) = Q(Aw, Bz, Bz, t)$$

which is a contradiction. Therefore $Ax = By$, $teAx = Sx = By = Ty$.

Let there exist another point $z$ such that $Az = Sz$. Then by inequality (1) we have $z = Sz = By = Ty$. Therefore $Ax = Az$. And the unique point of coincidence of $A$ and $S$. Then by Lemma 2.12 we have $w$ is the common fixed point of $A$ and $S$. Similarly there is a unique point $z \in X$ such that $Bz = Tz$. Assume that $w \neq x$. We have

$$Q(w, z, z, t) = Q(Aw, Bz, Bz, t)$$

which is a contradiction. Therefore $Ax = By$, $teAx = Sx = By = Ty$.

Let there exist another point $z$ such that $Az = Sz$. Then by inequality (1) we have $z = Sz = By = Ty$. Therefore $Ax = Az$. And the unique point of coincidence of $A$ and $S$. Then by Lemma 2.12 we have $w$ is the common fixed point of $A$ and $S$. Similarly there is a unique point $z \in X$ such that $Bz = Tz$. Assume that $w \neq x$. We have

$$Q(w, z, z, t) = Q(Aw, Bz, Bz, t)$$

which is a contradiction. Therefore $Ax = By$, $teAx = Sx = By = Ty$.

Let there exist another point $z$ such that $Az = Sz$. Then by inequality (1) we have $z = Sz = By = Ty$. Therefore $Ax = Az$. And the unique point of coincidence of $A$ and $S$. Then by Lemma 2.12 we have $w$ is the common fixed point of $A$ and $S$. Similarly there is a unique point $z \in X$ such that $Bz = Tz$. Assume that $w \neq x$. We have

$$Q(w, z, z, t) = Q(Aw, Bz, Bz, t)$$

which is a contradiction. Therefore $Ax = By$, $teAx = Sx = By = Ty$.
Hence \( z = w \) and \( z \) is a common fixed point of \( A, B, S \) and \( T \).

To prove uniqueness, let \( z' \) be another common fixed point of \( S \) and \( T' \).
Let \( z \neq z' \). We have
\[
Q(z',z,z,kt) = Q(Az',Bz,Bz,kt)
\]
\[
\geq \varphi(\{Q(Sz',Tz,Tz,t),Q(Sz',Bz,Tz,t), Q(Bz,Tz,Tz,t),Q(Az',Tz,Tz,t),Q(Az',Tz,Bz,kt)\})
\]
\[
= \varphi(\{Q(z',z,z,t),Q(z',z,z,t), Q(z',z,z,t),Q(z',z,z,t)\})
\]
\[
= \varphi(\{Q(z',z,z,t)\}).
\]
\[
> Q(z',z,z,t).
\]

Which is a contradiction. Hence \( z = z' \) is a unique common fixed point of \( A, B, S \) and \( T \).

**Theorem 3.2:** Let \((X, Q, *)\) be a complete \( Q \)-fuzzy metric space and let \( A, B, S \) and \( T \) be self mappings of \( X \). Let the pair \([A, S]\) and \([B, T]\) be occasionally weakly compatible (owc). If there exist a \( k \in (0, 1) \) such that for every \( x, y, z \in X \) and \( t > 0 \)
\[
Q(Ax, By, Bz, kt) \geq \min\{Q(Sx, Ty, Tz, t), Q(Sx, Ty, Tz, t), Q(Ax, Ty, Tz, t) + Q(Ax, Ty, Bz, t)/2\} \quad (2)
\]
Then there exists a unique common fixed point of \( A, B, S \) and \( T \).

Proof: The pair of self mappings \([A, S]\) and \([B, T]\) be occasionally weakly compatible (owc). So there exist points \( x, y \in X \) such that \( Ax = Sy \) and \( By = Ty \). First claim that \( Ax = By \). If not by inequality (2)
\[
Q(Ax, By, By, kt) \geq \min\{Q(Sx, Ty, Ty, t), Q(Sx, Ty, Ty, t), Q(Ax, Ty, Ty, t) + Q(Ax, Ty, By, t)/2\}
\]
\[
= \min\{Q(Ax, By, By, t), Q(Ax, By, By, t), Q(By, By, By, t), Q(Ax, By, By, t) + Q(Ax, By, By, t)/2\}
\]
\[
= Q(Ax, By, By, t)
\]

which is a contradiction. Therefore \( Ax = By \). ie \( Ax = Sy = By = Ty \).
Let there exist another point \( z \) such that \( Az = Sz \). Then by inequality (2) we have \( Az = Sz = By = Ty \). Therefore \( Ax = Az \). ie \( w = Ax = Sx \) is the unique point of coincidence of \( A \) and \( S \). Then by Lemma 2.12 \( w \) is the common fixed point of \( A \) and \( S \). Similarly there is a unique point \( z \in X \) such that \( z = Bz = Tz \).
Assume that \( w \neq z \). We have
\[
Q(w, z, z, kt) = Q(Aw, Bz, Bz, t)
\]
\[
\geq \min\{Q(Sw, Tz, Tz, t), Q(Sw, Bz, Tz, t), Q(Bz, Tz, Tz, t), Q(Aw, Tz, Tz, t) + Q(Aw, Tz, Bz, t)/2\}
\]
\[
= \min\{Q(w, z, z, t), Q(w, z, z, t), Q(z, z, z, t), Q(z, z, z, t) + Q(w, z, z, t)/2\}
\]
\[
= Q(w, z, z, t).
\]
Hence \( z = w \) and \( z \) is a common fixed point of \( A, B, S \) and \( T \).
To prove uniqueness, let \( z' \) be another common fixed point of \( A, B, S \) and \( T \). If \( z \neq z' \) we have
\[
Q(z', z, z, kt) = Q(Az', Bz, Bz, t)
\]
\[
\geq \min\{Q(Sz', Tz, Tz, t), Q(Sz', Bz, Tz, t), Q(Bz, Tz, Tz, t), Q(Az', Tz, Tz, t) + Q(Az', Tz, Bz, t)/2\}
\]
\[
= \min\{Q(z', z, z, t), Q(z', z, z, t), Q(z, z, z, t), Q(z, z, z, t) + Q(z', z, z, t)/2\}
\]
\[
= Q(z', z, z, t).
\]
Which is a contradiction. Hence \( z = z' \). ie \( z \) is a unique common fixed point of \( A, B, S \) and \( T \).

**Example**
Consider \( X = [1, \infty) \) and \((X, Q, *)\) is a complete \( Q\)-fuzzy metric space defined by
\[
Q(x, y, z, t) = \frac{t}{t + G(x, y, z)}.
\]
Here \((X, G)\) is a \( G \)-metric space defined by \( G(x, y, z) = d(x, y) + d(y, z) + d(z, x) \), \( d \) is usual metric on \( X \). \( A, B, S \) and \( T \) are self mappings of \( X \) given by \( Ax = x; Bx = x^2; Sx = x^3; Tx = x^4 \). Clearly the \( A, S \) and \( B, T \) are occasionally weakly compatible: \([0, 1]^2 \to [0, 1]\) such that
\[
\varphi(x, y, z, s, t) = \max\{x, y, z, s, t\} \quad \text{for } 0 < t < 1
\]
Here \( A, B, S \) and \( T \) satisfies condition (1) of theorem 3.1. Hence the four mappings have unique common fixed point. 1 is the common fixed point of \( A, B, S \) and \( T \).

**IV. CONCLUSION**
Fixed point theory has many applications in several branches of science such as game theory, nonlinear programming, economics, theory of differential equations, etc. In this paper we prove common fixed point theorems for four occasionally weakly compatible self maps in generalized fuzzy metric space. To prove the results we use the concept of occasionally weakly compatible maps. While proving the results we take \( Q\)-fuzzy metric...
spaces, which is a generalization of fuzzy metric space. Our results presented in this paper generalize and improve some known results in fuzzy metric space.

ACKNOWLEDGMENT

The first author would like to express sincere gratitude to St Mary’s College, Thrissur and University of Calicut for giving facilities and support during the preparation of this work.

REFERENCES