On Construction of Codes by using Fuzzy Sets and Soft Sets

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Abstract:
The objective of this paper is to study the codes arising from Fuzzy sets and Soft sets. Properties of fuzzy linear codes and fuzzy cyclic codes are discussed by means of fuzzy linear space. P-fuzzy sets are considered as mapping from an arbitrary non-empty set $S$ into a partially ordered set $P$ which determines a binary block code $V$ of length $n$. At last, the codes developed by using soft sets, called soft codes (soft linear codes) are discussed with examples.

AMS Mathematics Subject Classification (2010) : 03E72, 06D72, 11T71.

Keywords: Fuzzy linear code, P-fuzzy set, Soft set, Soft linear code.

1. INTRODUCTION
Coding theory is concerned with reliability of communication over noisy channel. Algebraic codes [15] are used for data compression, error correction and for network coding. The theory of error-correcting codes was first introduced by Claude Shannon in 1948 and then gradually developed by time to time by different researchers. Messages in the form of bit strings are encoded by translating them into longer bit strings, called codeword. A set of codeword is called a code. There are many types of codes which is important to its algebraic structures such as Linear block codes, Hamming codes, BCH codes and so on. The most common type of code is a linear code over the finite field $F_q$. More literature can be studied in [15, 16, 17, 19, 21].

Fuzzy sets were introduced by Lotfi A. Zadeh [3] as an extension of crisp set (classical set). Fuzzy set theory permits the gradual assessment of the membership of elements in a set which is described in the interval $[0, 1]$. It can be used in a wide range of domains where the information is partial and vague. Fuzzy set perhaps is the most suitable framework to model uncertain data. Fuzzy sets have a several applications in the areas such as signal processing, decision making, control theory, pattern recognition, computer version and so on. The complexities of modeling uncertain data are the main problem in engineering, environmental science, social sciences, health and medical sciences etc. The fuzzy set theory [2], probability theory, rough set theory [10] etc. are well known and useful mathematical tools which describes uncertainty but each of them has its own limitation pointed out by Molodstov. Therefore, Molodstov introduced the theory of soft sets [1] to model vague and uncertain information. A soft set is a parameterized collection of subsets of a universe of discourse. A huge amount of literature can be seen in [1, 2, 3, 5, 9, 10, 11].

In this paper, codes arising from fuzzy linear space and $P$-fuzzy sets are discussed. $P$-fuzzy sets are considered as mapping from an arbitrary non-empty set $S$ into a partially ordered set $P$, which determines a binary block-code $V$ of length $n$. Further, soft codes (soft linear codes) are discussed through the application of soft sets which is an approximated collection of codes.

The paper is organized as follows: In section 2, basic concepts of fuzzy sets, soft sets and linear codes are presented. In section 3, codes arising from fuzzy linear space and $P$-fuzzy sets are discussed. In section 4, codes from soft sets, called soft codes (soft linear codes) are presented along with their fundamental definitions and examples.

2. Preliminaries:
This section has two subsections. In the first section, we recapitulate some underlying definitions and basics of Fuzzy sets and Soft sets. The second section recalls all the basic notion of linear algebraic codes.

2.1. Fuzzy Sets and Soft Sets:

Definition 2.1.1: [3] A fuzzy set $A$ in $R$ (real line) is defined to be a set of ordered pairs,
\( A = \{(x, A(x)) \mid x \in \mathbb{R}\} \) where \( A(x) \) is called the membership function for the fuzzy set.

**Definition 2.1.2:**[3] The \( \alpha \)-cut of \( \alpha \)-level set of fuzzy set \( A \) is a set consisting of those elements of the universe \( X \) whose membership values exceed the threshold level \( \alpha \).

That is \( A_\alpha = \{x \in X \mid A(x) \geq \alpha\} \).

**Definition 2.1.3:**[14] Let \( S \) be a non-empty set and \((P, \leq)\) a partially ordered set.

Any function \( A : S \rightarrow P \) is a \( P \)-fuzzy set on \( S \).

Also, for \( p \in P \), \( A : S \rightarrow \{0,1\} \)

So that for \( x \in S \), \( A_p(x) = 1 \) if \( A(x) \geq p \). Here \( A_p(x) \) is a characteristic function of a \( p \)-level subset (or a \( p \)-cut). That is \( A_p = \{x \in S \mid A(x) = 1\} \).

**Remark 1:**[14] Let \( A : S \rightarrow P \) be a \( P \)-fuzzy set on \( S \), and \( \sim \) a binary relation on \( P \), then for \( p, q \in P \), \( p \sim q \) if and only if \( A_p = A_q \).

Obviously, \( \sim \) is an equivalence relation on \( P \).

**Lemma 1:**[14] Let \( A : S \rightarrow P \) be a fuzzy set. For every \( x \in S \), if \( A(x) = p \), then \( p \) is a supremum of the class to which it belongs, that is, \( p = \bigvee \{p\} \sim \).

**Theorem 1:**[14] (Decomposition of \( P \)-fuzzy sets)

If \( A : S \rightarrow P \) is a \( P \)-fuzzy set on \( S \), for \( x \in S \), \( A(x) = \bigvee \{p \in P \mid A_p(x) = 1\} \).

That is, the supremum of the right exists in \((P, \leq)\) for every \( x \in S \), and is equal to \( A(x) \).

**Definition 2.1.4:**[18] Let \( V_n \) be a \( n \)-dimensional linear space over a field \( F_q \), \( A \) a fuzzy subset of \( V_n \), if for any \( x, y \in V_n \), \( \alpha \in F_q \), we have

\[
\begin{align*}
(1) & \quad A(x + y) \geq \min \{A(x), A(y)\} ; \quad (2) \quad A(\alpha x) \geq A(x) \\
& \quad \text{Then,} \quad A \text{ the fuzzy linear subspace of} \ V_n \text{ on} \ F_q .
\end{align*}
\]

**Definition 2.1.5:**[1] A pair \((F, A)\) is called a soft set over \( U \), where \( F \) is mapping given by \( F : A \rightarrow P(U) \). Here \( U \) refers to an initial universe and \( E \) to be a set of parameters. \( P(U) \) denote the power set of \( U \) and \( A \subseteq E \).

A soft set over \( U \) is a parameterized family of subsets of the universe \( U \). For \( e \in A \), \( F(e) \) is considered to be set of \( e \)-elements of the soft set \((F, A)\) or as the set of \( e \)-approximate elements of the soft set.

**Definition 2.1.6:**[1] Let \((F, A)\) and \((G, B)\) be the two soft sets over a common universe \( U \), then \((F, A)\) is called a soft subset of \((G, B)\) denoted by \((F, A) \subseteq (G, B)\) if,

1. \( A \subseteq B \)
2. \( \forall e \in A, F(e) \text{ and } G(e) \) are identical approximations.

### 2.2 Linear Algebraic Codes:

**Definition 2.2.1:**[21] A code \( C \) is any non-empty subset of \( F_q^n \). The code \( C \) is called linear, if it is an \( F_q \)-linear subspace of \( F_q^n \). The number \( n \) is the length of the code.

**Definition 2.2.2:**[21] The Hamming distance \( d \) on \( F_q^n \) is given by \( d(x, y) := \min \{i \mid 1 \leq i \leq n, x_i \neq y_i\} \)

where \( x = (x_1, \ldots, x_n) \) and \( y = (y_1, \ldots, y_n) \).

**Definition 2.2.3:**[21] The minimum distance of a code \( C \subseteq F_q^n \) is given by \( d(C) := \min \{d(x, y) \mid x, y \in C, x \neq y\} \).

**Definition 2.2.4:**[21] The linear code \( C \) is called linear \([n, k]-\)code, if \( \dim(C) = k \).

**Definition 2.2.5:**[21] Let \( C \) be a linear \([n, k]-\)code. Let \( G \) be a \( k \times n \) matrix whose rows forms a basis of \( C \). Then \( G \) is called generator matrix of the code \( C \).

**Definition 2.2.6:**[16][21] Let \( C \) be an \([n, k]-\)code over \( F_q \). Then the dual code of \( C \) is denoted as
Let \( A \) be a fuzzy linear subspace, if for any \( \alpha \in [0,1] \), \( A_\alpha \neq \phi \). \( A_\alpha \) is a linear subspace of \( V_n \).

**Definition 3.3:** A fuzzy linear subspace \( A \) of \( V_n \) is called a fuzzy cyclic code, if for any \( (a_0, a_1, ..., a_{n-1}) \in V_n \), we have \( A((a_{n-1}, a_0, ..., a_{n-2})) \supseteq A((a_0, a_1, ..., a_{n-1})) \).

**Lemma 3:** \( A \) is a fuzzy cyclic code, iff for any \( \alpha \in [0,1] \), \( A_\alpha \neq \phi \). \( A_\alpha \) is a cyclic code.

**Theorem 3:** Let \( V = \{v_1, ..., v_n\} \subseteq \{0,1\}^n \) be a binary block-code, such that for every \( i \) \( \in \{1, ..., n\} \), at least one codeword has a nonzero \( i^{th} \) coordinate. Then there is a \( P \)-fuzzy set which corresponds to \( V \) if and only if, for every \( i \) \( \in \{1, ..., n\} \), \( \forall (v \in V) (v(i) = 1) \in V \).

The Hamming distance \( d(x, y) \) between \( x \) and \( y \) from \( \{0,1\}^n \) is the number of coordinates in which \( x \) and \( y \) differ. That is, \( d(x, y) := \{i | x_i \neq y_i\} \), \( x, y \in \{0,1\}^n \).

The code distance of \( V \subseteq \{0,1\}^n \) is the minimum Hamming distance between two codewords in \( V \) denoted by \( d(V) \) and is defined as,

\[
d(V) := \min\{d(x, y) : x, y \in \{0,1\}^n, x \neq y\}.
\]
4. Codes from Soft Sets (Soft Linear Codes):

In this section the codes evolved from soft sets, called soft codes (soft linear codes) are discussed along with examples.

Definition 4.1: Let $F_q$ be a finite field and $V = F_q^n$ be a vector space over $F_q$, where $n$ is a positive integer. Let $P(V)$ be the power set of $V$ and $(F, A)$ be a soft set over $V$. Then $(F, A)$ is called soft linear code over $V$ if and only if, $F(e)$ is a subspace of $V$ which is a linear code.

Example 1. Let $F_q = F_2$ and $V = F_2^3$ is a vector space over $F_2$ and let $(F, A)$ be a soft set over $V = K_2^3$. Then $(F, A)$ is a soft linear code over $V = K_2^3$, where $F(e_1) = \{000, 111\}$, $F(e_2) = \{000, 101, 110, 111\}$.

Definition 4.2: Let $(F, A)$ be a soft code over $V = F_q^n$. Then $D_s$ is called soft dimension of $(F, A)$ if $D_s = \{\dim(F(e)), \forall e \in A\}$.

Example 2. Let $(F, A)$ be a soft code defined as in above Example 1. Then the soft dimension is given by $D_s = \{\dim(F(e_1)) = 1, \dim(F(e_2)) = 2\} = \{1, 2\}$.

Definition 4.3: A soft linear code $(F, A)$ over $F_q^n$ of soft dimension $D_s$ is called soft linear $[n, D_s]$-code.

Definition 4.4: Let $(F, A)$ be a soft code over $V = F_q^n$. The soft minimum distance of $(F, A)$ is denoted by $S_d(F, A)$ and is defined to be $S_d(F, A) = d(F(e))$, where $d(F(e))$ is the minimum distance of the code $F(e)$; for all $e \in A$.

Example 3. Let $F_q = F_2$ be a finite field and $V = F_2^3$ is a vector space over $F_2$ and let $(F, A)$ be a soft set over $V = F_2^3$. Then clearly $(F, A)$ is a soft code over $V = F_2^3$, where $F(e_1) = \{000, 111\}$, $F(e_2) = \{000, 101, 110, 111\}$. The minimum distance of the code $F(e_1) = 3$ and $F(e_2) = 2$. Thus the soft minimum distance of the soft code $(F, A)$ is given as $S_d(F, A) = \{d(F(e_1)) = 3, d(F(e_2)) = 2\} = \{3, 2\}$.

Definition 4.5: Let $(F, A)$ be a soft linear $[n, D_s]$-code. Let $G_s$ be the $n$-matrix whose elements are the generator matrices of the soft code $(F, A)$ corresponding to each $e \in A$ where $n$ is the number of parameters in $A$.

The matrix $G_s$ is termed as the soft generator matrix of the soft linear code $(F, A)$.

Definition 4.6: Let $(F, A)$ be a soft $[n, D_s]$-code over the field $F_q$ and let the vector space $V = F_q^n$. Then the soft dual code of $(F, A)$ is defined to be $\overline{(F, A)} = F(e)^\perp : F(e)^\perp$ is the dual code of $F(e)$, for all $e \in A$.

Example 4. Let $F_q = F_2$ be a finite field and $V = F_2^3$ is a vector space over $F_2$ and let $(F, A)$ be a soft set over $V = F_2^3$. Then clearly $(F, A)$ is a soft code over $V = F_2^3$, where $F(e_1) = \{000, 111\}$ and $F(e_2) = \{000, 101, 110, 111\}$.

Then the soft dual code of $(F, A)$ is denoted as $\overline{(F, A)} = F(e)^\perp$ where $F(e_1)^\perp = \{000, 110, 101, 011\}$ and $F(e_2)^\perp = \{000, 111\}$.

Definition 4.7: A soft linear code $(F, A)$ in $V = F_q^n$ over the field $F_q$ is called soft self-dual code if $\overline{(F, A)} = (F, A)$.
Definition 4.8: [13] A soft linear code \((F, A)\) in 
\(V = F^n_q\) over the field \(F_q\) is called complete-soft code, if for all \(e \in A\), the dual of \(F(e)\) also exists in \((F, A)\).

Example 5. Let \(q = 2\) be a finite field and 
\(V = F_2^3\) is a vector space over \(F_2\) and let \((F, A)\) be a soft set over \(V = F_2^3\). Then \((F, A)\) is a soft code over \(V = F_2^3\), where \(F(e_1) = \{000,111\}\) and \(F(e_2) = \{000,110,101,011\}\). Then \((F, A)\) is a complete-soft code. Since the dual of \(F(e_1)\) and \(F(e_2)\) and also the dual of \(F(e_2)\) is \(F(e_1)\).

Theorem 4. [13] All complete-soft codes are trivially soft codes but the converse is not true in general.

5. Conclusion:

In this paper, codes from Fuzzy sets and Soft sets are discussed at the basic level. The advantage of soft codes (soft linear code) is that it can send \(n\)-messages to \(n\)-persons.

Acknowledgments:

The author is highly grateful to the anonymous referees and Mr. Ravi Rewaskar, Faculty, PHCET, Panvel, Maharashtra, India, for their valuable comments and constructive suggestions which help me to improve the present form of this paper.

References: