Idempotent generators of quadratic residue cyclic codes of length $4p^nq^m$

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Abstract

We consider the ring $R_{4p^mq} = GF(1)[x]/(x^{4p^mq} - 1)$, where $p, q$ and $\ell$ be distinct odd primes ($\ell$ is of the type 4k+1), $\ell$ is quadratic residue modulo $2p^n$ as well as modulo $2q^m$ ($n, m \geq 1$) with gcd $(\phi(2p^n)/2, \phi(2q^m)/2) = 1$. Explicit expressions for all the $8mn+8n+8m+4$ primitive idempotents are obtained.

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1. Introduction

Let $GF(\ell)$ be a field of odd prime order $\ell$ of the type 4k+1. Let $\eta \geq 1$ be an integer with gcd $(\ell, \eta) = 1$. Let $R_{\eta} = GF(\ell)[x]/(x^\eta - 1)$. The minimal cyclic codes of length $\eta$ over $GF(\ell)$ are ideals of the ring $R_{\eta}$ generated by the primitive idempotents. Arora and Pruthi [1, 2] obtained the primitive idempotents in $R_{\eta}$ for $\eta = 2, 4, p^n, 2p^n$ where $p$ is an odd prime and $\ell$ is primitive root mod $\eta$. When $\eta = p^nq$ where $p$, $q$ are distinct odd primes and $\ell$ is a primitive root mod $p^n$ and $q$ both with gcd $(\phi(p^n)/2, \phi(q)/2) = 1$, the primitive idempotent in $R_{\eta}$ have been obtained by, G.K.Bakshi and Madhu Raka [4]. $\eta = p^nq^m$, where $p, q$ and $\ell$ be distinct odd primes ($\ell$ is of the type 4k+1), $\ell$ is quadratic residue modulo $p^n$ as well as modulo $q^m$ $(n, m \geq 1)$ with gcd $(\phi(p^n)/2, \phi(q^m)/2) = 1$, the primitive idempotent in $R_{\eta}$ have been obtained by, Ranjeet Singh and Manju Pruthi [5]. $\eta = 2p^nq^m$, where $p, q$ and $\ell$ be distinct odd primes, $\ell$ is quadratic residue modulo $2p^n$ as well as modulo $q^m$ $(n, m \geq 1)$ with gcd $(\phi(p^n)/2, \phi(q^m)/2) = 1$, the primitive idempotent in $R_{\eta}$ have been obtained by, Ranjeet Singh [6]. In this paper, we consider the case when $\eta = 4p^nq^m$ where $p, q$ and $\ell$ be distinct odd primes ($\ell$ is of the type 4k+1), $\ell$ is quadratic residue modulo $2p^n$ as well as modulo $2q^m$ $(n, m \geq 1)$ with gcd $(\phi(2p^n)/2, \phi(2q^m)/2) = 1$. We obtain explicit expressions for all the $8mn+8n+8m+4$ primitive idempotents in $R_{4p^nq^m}$ (see theorem 2.3).

2. Primitive Idempotents in $R_{4p^nq^m} = GF(1)[x]/(x^{4p^nq^m} - 1)$
2.1. For \( 0 \leq s \leq \eta - 1 \), let \( \mathcal{C}_s = \{ s, sI, sI^2, \ldots, sI^{t_s-1} \} \), where \( t_s \) is the least positive integer such that \( sI^{t_s} \equiv s \pmod{\eta} \) be the cyclotomic coset containing \( s \), if \( \alpha \) denotes a primitive \( \eta \)th root of unity in some extension field of \( GF(\ell) \), then the polynomial \( M^s(x) = \prod_{i \in \mathcal{C}_s} (x - \alpha^i) \) is the minimal polynomial of \( \alpha^s \) over \( GF(\ell) \).

Let \( M_s \) be the minimal ideal in \( R_\eta \) generated by \( \frac{x^{\frac{M}{2}} - 1}{M^s(x)} \) and \( \theta_s \) be the primitive idempotent of \( M_s \), then we know by (Theorem 1, [4]) the primitive idempotent \( \theta_s \) corresponding to the cyclotomic coset \( \mathcal{C}_s \) containing \( s \) in \( R_{4p^q} \) is given by \( \theta_s = \sum_{i \in \mathcal{C}_s} \epsilon^i \), where \( s_i = \frac{1}{4p^{q_m}} \sum_{j \in \mathcal{C}_s} \alpha^{-ij} \forall i \geq 0 \). Thus to describe \( \theta_s \), it becomes necessary to compute \( \epsilon_i \). To compute \( \epsilon_i \) numerically, we consider the case when \(-C_1 = C_{3ab}\) and we get that

\[
\epsilon_i = \frac{1}{4p^{q_m}} \sum_{j \in \mathcal{C}_s} \alpha^{-ij} = \frac{1}{4p^{q_m}} \sum_{j \in \mathcal{C}_s} \alpha^{ij} \forall i \geq 0.
\]

**Lemma 2.2.** Let \( p, q, l \) be distinct odd primes \((l \) is of the type \( 4k+1 \)) and \( \ell \) is quadratic residue modulo \( 2p^n \) as well as modulo \( 2q^m \) \((n, m \geq 1) \) with \( \gcd(\phi(2p^n)/2, \phi(2q^m)/2) = 1 \). Then

\[
0(l) \left( \frac{4p^{q_m} - 1}{8} \right), \text{ for all } k, 0 \leq j \leq n - 1, 0 \leq k \leq m - 1.
\]

**Theorem 2.3.** The \( 8mn+8n+m+4 \) primitive idempotents corresponding to cyclotomic cosets

\[
\begin{align*}
&c_{o1}c_{p^nq_m}, c_{2p^nq_m}, c_{3p^nq_m}, c_{p^nq'_1}, c_{2p^nq'_1}, c_{3p^nq'_1}, c_{4p^nq'_1}, c_{2bp^nq'_1}, c_{3bp^nq'_1}, c_{4bp^nq'_1} \\
&c_{4bp^nq'_1}, c_{2p^nq_m}, c_{3p^nq_m}, c_{4p^nq_m}, c_{2ap^nq_m}, c_{3ap^nq_m}, c_{4ap^nq_m}, c_{2bp^nq_m}, c_{3bp^nq_m}, c_{4bp^nq_m} \\
&c_{2ap^nq_m}, c_{3ap^nq_m}, c_{4ap^nq_m} \\
&0 \leq j \leq n - 1, 0 \leq k \leq m - 1 \text{ in } R_{4p^nq_m} \text{ are}
\end{align*}
\]

(i) \( \theta_s(x) = \frac{1}{4p^{q_m}} \left( 1 + x + x^2 + \ldots + x^{4p^{q_m} - 1} \right) \).

(ii) \( \theta_s(x) = \frac{1}{4p^{q_m}} \left( \sum_{(i,j)=(0,0)}^{(n,m)} (\sigma_{4(i,j)}(x) + \sigma_{4a(i,j)}(x) - \sigma_{2(i,j)}(x) - \sigma_{2a(i,j)}(x)) + \right) \)

\[
\sum_{(i,j)=0}^{(n,m)} (\sigma_{4ab(i,j)}(x) + \sigma_{4b(i,j)}(x) - \sigma_{2b(i,j)}(x) - \sigma_{2ab(i,j)}(x))(x) + \right) \]

\[
\sum_{(i,j)=0}^{(n,m)} \left( (\sigma_{3(i,j)}(x) + \sigma_{3a(i,j)}(x) - \sigma_{i(i,j)}(x) - \sigma_{ai(i,j)}(x)) \right) + \right) \]

\[
+ \sum_{(i,j)=0}^{(n,m)} \left( (\sigma_{3ab(i,j)}(x) + \sigma_{3b(i,j)}(x) - \sigma_{ab(i,j)}(x) - \sigma_{b(i,j)}(x)) \right) \right)
\]
(iii) Replacing $-\sigma_{2(i,r)}(x)$ by $\sigma_{2(i,r)}(x)$, $\sigma_{3(i,r)}(x)$ by $-\sigma_{3(i,r)}(x)$ and $-\sigma_{(i,r)}(x)$ by $-\sigma_{(i,r)}(x)$ and we get the required expression for $\theta_{3p^mq^m}(x)$.

(iv) Replacing $-\sigma_{2(i,r)}(x)$ by $\sigma_{2(i,r)}(x)$, $\sigma_{3(i,r)}(x)$ by $-\sigma_{3(i,r)}(x)$ and $-\sigma_{(i,r)}(x)$ by $-\sigma_{(i,r)}(x)$ and we get the required expression for $\theta_{3p^mq^m}(x)$.

(v) For $0 \leq k \leq m - 1,$

$$\theta_{p^mq^m}(x) = \frac{1}{4p^mq^m} \left( \sum_{(i,r)=(0,0)}^{(n-1,m-k-1)} \eta_i^n \sigma_{(i,r)}(x) + \sigma_{4ab(i,r)}(x) + \sigma_{3ab(i,r)}(x) + \sigma_{1a(i,r)}(x) + \sigma_{2a(i,r)}(x) + \sigma_{3b(i,r)}(x) + \sigma_{2b(i,r)}(x) \right)$$

$$+ \frac{\phi(q^{m-k})}{8p^mq^m} \left\{ \sum_{(i,r)=(0,0)}^{(n-1,m-1)} \sigma_{4(i,r)}(x) + \sigma_{4ab(i,r)}(x) - \sigma_{2(i,r)}(x) - \sigma_{2ab(i,r)}(x) \right\} \right.$$
\[ \begin{align*}
&+ \sum_{(i,r) \in \mathbb{D}(n,m)} \left[ \sigma_{4a(i,r)}(x) - \sigma_{2a(i,r)}(x) + i(\sigma_{3a(i,r)}(x) - \sigma_{a(i,r)}(x)) \right] \\
&+ \left( \sigma_{4(n,m)}(x) - \sigma_{2(n,m)}(x) + i(\sigma_{3(n,m)}(x) - \sigma_{2(n,m)}(x)) \right)
\end{align*} \]

(vi) Similarly (v), replacing \( \eta_0^* \) by \( \xi_0^* \) and \( \xi_1^* \) by \( \xi_0^* \) and \( \eta_1^* \) by \( \xi_1^* \), we get the required expression for \( \theta_{4p^qk}(x) \).

(vii) Similarly (v) replacing \( \xi_1^* \) by \( \xi_0^* \) and \( \xi_0^* \) by \( \xi_1^* \), \( \eta_1^* \) by \( \xi_1^* \) and \( \eta_1^* \) by \( \xi_1^* \), we get the required expression for \( \theta_{3p_qk}(x) \).

(viii) Similarly (v) replacing \( \eta_0^* \) by \( \xi_0^* \) and \( \xi_1^* \) by \( \xi_1^* \), \( \eta_1^* \) by \( \xi_1^* \) and \( \xi_0^* \) by \( \xi_0^* \), we get the required expression for \( \theta_{bp^qk}(x) \).

(ix) Similarly (v), replacing \( \eta_1^* \) by \( \eta_1^* \) and \( \xi_0^* \) by \( \xi_0^* \) in \( \theta_{b^p_qk}(x) \) and we get the required expression for \( \theta_{b^p_qk}(x) \).

(x) Similarly (vi), replacing \( \xi_0^* \) by \( \xi_0^* \) and \( \xi_1^* \) by \( \xi_1^* \) in \( \theta_{2p^qk}(x) \) and we get the required expression for \( \theta_{2p^qk}(x) \).

(xi) Similarly (vii), replacing \( \eta_0^* \) by \( \eta_1^* \) and \( \xi_0^* \) by \( \xi_1^* \) and \( \xi_1^* \) by \( \xi_0^* \) in \( \theta_{3p^qk}(x) \) and we get the required expression for \( \theta_{3p^qk}(x) \).

(xii) Similarly (viii), replacing \( \xi_1^* \) by \( \xi_1^* \) and \( \xi_0^* \) by \( \xi_0^* \) in \( \theta_{4p^qk}(x) \) and we get the required expression for \( \theta_{4p^qk}(x) \).

(xiii) For \( 0 \leq j \leq n-1 \)
\[ \theta_{p^q}^{n-j}(x) = \frac{1}{4p^q} \sum_{r=0}^{m-1} \left[ \eta_0(\sigma_{n-j-1,r}(x) + \sigma_{3(n-j-1,r)}(x) + \sigma_{2b(n-j-1,r)}(x) + \sigma_{3ab(n-j-1,r)}(x)) \right] \]
\[+ \sum_{r=0}^{m-1} \xi_0(\sigma_{2n-j-1,r}(x) + \sigma_{2a(n-j-1,r)}(x) + \sigma_{2ab(n-j-1,r)}(x) + \sigma_{4ab(n-j-1,r)}(x)) \]
\[+ \sum_{r=0}^{m-1} \eta_1(\sigma_{a(n-j-1,r)}(x) + \sigma_{b(n-j-1,r)}(x) + \sigma_{3a(n-j-1,r)}(x) + \sigma_{3b(n-j-1,r)}(x)) \]
\[+ \sum_{r=0}^{m-1} \xi_1(\sigma_{2a(n-j-1,r)}(x) + \sigma_{4a(n-j-1,r)}(x) + \sigma_{2ab(n-j-1,r)}(x) + \sigma_{4b(n-j-1,r)}(x)) \]
\[+ \phi(\sigma_{n-j}(x) - \sigma_{2(n-1,j)}(x) + i(\sigma_{3(n-1,j)}(x) - \sigma_{(n-1,j)}(x)) \right] \]
We get the required expression by

\[
\sigma_{4b(i,r)}(x) - \sigma_{2b(i,r)}(x) + i(\sigma_{3b(i,r)}(x) - \sigma_{b(i,r)}(x))
\]

and \(\xi_0\) by \(\xi_1\), \(\eta_1\) by \(\xi_1\), \(\eta_0\) by \(\xi_0\), \(\xi_1\) by \(\xi_0\), \(\eta_1\) by \(\eta_0\), we get the required expression for \(\vartheta_{2p^2q^m}(x)\).

(xvi) Similarly (xiii), Replacing \(-\sigma_2(i,r)(x)\) by \(\sigma_2(i,r)(x)\), \(\partial\sigma_3(i,r)(x)\) by \(-\sigma_3(i,r)(x)\) and \(-\partial\sigma(i,r)(x)\) by \(-\sigma(i,r)(x)\) and \(\xi_0\) by \(\xi_1\), \(\eta_1\) by \(\xi_1\), \(\eta_0\) by \(\xi_0\), \(\xi_1\) by \(\xi_0\), \(\eta_1\) by \(\eta_0\), we get the required expression for \(\vartheta_{4p^2q^m}(x)\).

(xvii) Similarly (xiii) Replacing \(\eta_0\) by \(\eta_1\), \(\xi_1\) by \(\xi_0\), we get the required expression for \(\vartheta_{ap^2q^m}(x)\).

(xviii) Similarly (xiv) Replacing \(\xi_0\) by \(\xi_1\), \(\xi_1\) by \(\xi_0\), we get the required expression for \(\vartheta_{3ap^m}(x)\).

(xix) Similarly (xv) Replacing \(\eta_0\) by \(\eta_1\), \(\xi_0\) by \(\xi_1\), \(\eta_1\) by \(\eta_0\), \(\xi_1\) by \(\xi_0\), we get the required expression for \(\vartheta_{3ap^m}(x)\).

(xx) Similarly (xvi) Replacing \(\xi_0\) by \(\xi_1\), \(\xi_1\) by \(\xi_0\), we get the required expression for \(\vartheta_{3ap^m}(x)\).

(xxii) For \(0 \leq j \leq n-1\), \(0 \leq k \leq m-1\)
\[
\theta_{p,q}^{i,j,k}(x) = \frac{1}{4p^q} \frac{1}{m^k} \sum_{(i,j,r,k) = (0,0)}^{(n-1,m-k-1)} \left[ C_{i,j,r,k}(\sigma_{l,r,j}(x) + \sigma_{a,b}(j,r)(x)) + B^*_{i,j,r,k}(\sigma_{2,l,j}(x) + \sigma_{2,b}(j,r)(x))\right]
\]

\[
A_{i,j,r+k}(\sigma_{3,l,j}(x) + \sigma_{3,a,b}(j,r)(x)) + D_{i,j,r+k}(\sigma_{4,l,j}(x) + \sigma_{4,a,b}(j,r)(x))
\]

\[
+ \frac{1}{p^q} \sum_{(i,j,r,k) = (0,0)}^{(n-1,m-k-1)} \left[ D_{i,j,r,k}(\sigma_{a,l,j}(x) + \sigma_{b,l,j}(x)) + A^*_{i,j,r+k}(\sigma_{2,a,l,j}(x) + \sigma_{2,b,l,j}(x))\right]
\]

\[
+ B_{i,j,r+k}(\sigma_{3,a,l,j}(x) + \sigma_{3,b,l,j}(x)) + C^*_{i,j,r+k}(\sigma_{4,a,l,j}(x) + \sigma_{4,b,l,j}(x))
\]

\[
+ \phi p^{n-1} q^{m-1} \phi q^{-n-1} m^{-1} [\sum_{(i,j,r,k) = (n-1,j,m-k-1)}^{(n-1,m-k-1)} \left[ (\sigma_{a(i,m,k-1)}(x) + \sigma_{a,b}(j,i,m,k-1)(x) + \sigma_{3a(i,m,k-1)}(x) + \sigma_{4a(i,m,k-1)}(x))\right]
\]

\[
+ \sum_{(i,j,r,k) = (n-1,j,m-k-1)}^{(n-1,m-1)} \left[ (\sigma_{b(i,m,k-1)}(x) + \sigma_{b,a}(j,i,m,k-1)(x) + \sigma_{3b(i,m,k-1)}(x) + \sigma_{4b(i,m,k-1)}(x))\right]
\]

\[
+ \sum_{(i,j,r,k) = (n-1,j,m-k-1)}^{(n-1,m-k)} \left[ (\sigma_{ab(i,m,k-1)}(x) + \sigma_{2ab}(j,i,m,k-1)(x) + \sigma_{3ab}(j,i,m,k-1)(x) + \sigma_{4ab}(j,i,m,k-1)(x))\right]
\]

\[
+ \phi 4p^{n-1} q^{m-1} [\sum_{(i,j,r,k) = (n-1,j,m-k)}^{(n-1,m-1)} \left[ (\sigma_{4,l,j}(x) - \sigma_{2,l,j}(x) + \sigma_{a,b}(j,r)(x))\right]
\]

\[
+ \sum_{(i,j,r,k) = (n-1,j,m-k)}^{(n-1,m-1)} \left[ (\sigma_{4a,l,j}(x) - \sigma_{2a,l,j}(x) + \sigma_{a,b}(j,r)(x))\right]
\]

\[
+ \sum_{(i,j,r,k) = (n-1,j,m-k)}^{(n-1,m-1)} \left[ (\sigma_{4b,l,j}(x) - \sigma_{2b,l,j}(x) + \sigma_{a,b}(j,r)(x))\right]
\]

\[
+ \sum_{(i,j,r,k) = (n-1,j,m-k)}^{(n-1,m-1)} \left[ (\sigma_{4ab,l,j}(x) - \sigma_{2ab,l,j}(x) + \sigma_{a,b}(j,r)(x))\right]
\]

\[
\]
(n,m-1) \sum_{(i,r)=(n,m-k)}^{(n,m-1)} (\sigma_{4(i,r)}(x) - \sigma_{2(i,r)}(x) + i(\sigma_{3(i,r)}(x) - \sigma_{(i,r)}(x))) + \\
(n-1,m) \sum_{(i,r)=(n-j,m)}^{(n-1,m)} (\sigma_{4(i,r)}(x) - \sigma_{2(i,r)}(x) + i(\sigma_{3(i,r)}(x) - \sigma_{(i,r)}(x))) + \\
(n-1,m) \sum_{(i,r)=(n-j,m)}^{(n-1,m)} (\sigma_{4(n,m)}(x) - \sigma_{2(n,m)}(x) + i(\sigma_{3(n,m)}(x) - \sigma_{(n,m)}(x)))}

(xxii) On the similar lines as in (xii), we can find \theta_{2p/q}(x) by replacing \mathcal{C}_{(i+j,r+k)} = \mathcal{A}_{(i+j,r+k)}^*.

\mathcal{B}_{(i+j,r+k)}^* = \mathcal{C}_{(i+j,r+k)}^* = \mathcal{A}_{(i+j,r+k)}^*, \quad \mathcal{D}_{(i+j,r+k)} = \mathcal{B}_{(i+j,r+k)}, \quad \mathcal{A}_{(i+j,r+k)}^* = \mathcal{D}_{(i+j,r+k)}^*.

A^*_{(i+j,r+k)} = \mathcal{A}_{(i+j,r+k)}^* = \mathcal{C}_{(i+j,r+k)}^* = \mathcal{D}_{(i+j,r+k)}^* + \phi(4p^n - iq^{m-k}) \quad \text{and vice versa in } \theta_{p/q}(x).

(xxiii) On the similar lines as in (xii), we can find \theta_{3p/q}(x) by replacing \mathcal{C}_{(i+j,r+k)} = \mathcal{A}_{(i+j,r+k)}^*.

\mathcal{B}_{(i+j,r+k)} = \mathcal{D}_{(i+j,r+k)} = \mathcal{B}_{(i+j,r+k)}^* = \mathcal{A}_{(i+j,r+k)}^*, \quad \mathcal{C}_{(i+j,r+k)}^* = \mathcal{D}_{(i+j,r+k)}^* + \phi(4p^n - iq^{m-k}) \quad \text{and vice versa in } \theta_{p/q}(x).

(xxiv) On the similar lines as in (xii), we can find \theta_{4p/q}(x) by replacing \mathcal{C}_{(i+j,r+k)} = \mathcal{C}_{(i+j,r+k)}^* = \mathcal{A}_{(i+j,r+k)}^* = \mathcal{B}_{(i+j,r+k)}.

\mathcal{B}_{(i+j,r+k)}^* = \mathcal{C}_{(i+j,r+k)}^* = \mathcal{A}_{(i+j,r+k)}^*, \quad \mathcal{D}_{(i+j,r+k)} = \mathcal{B}_{(i+j,r+k)}, \quad \mathcal{C}_{(i+j,r+k)}^* = \mathcal{D}_{(i+j,r+k)}^* + \phi(4p^n - iq^{m-k}) \quad \text{and vice versa in } \theta_{p/q}(x).

(xxv) On the similar lines as in (xii), we can find \theta_{3p/q} by replacing \mathcal{C}_{(i+j,r+k)} = \mathcal{D}_{(i+j,r+k)}.

\mathcal{B}_{(i+j,r+k)}^* = \mathcal{A}_{(i+j,r+k)}^*, \quad \mathcal{A}_{(i+j,r+k)} = \mathcal{B}_{(i+j,r+k)}, \quad \mathcal{D}_{(i+j,r+k)}^* = \mathcal{C}_{(i+j,r+k)}^* \quad \text{and vice versa in } \theta_{p/q}(x).

(xxvi) On the similar lines as in (xii), we can find \theta_{2p/q}(x) by replacing \mathcal{A}_{(i+j,r+k)}^* = \mathcal{B}_{(i+j,r+k)} = \mathcal{C}_{(i+j,r+k)}.

\mathcal{C}_{(i+j,r+k)}^* = \mathcal{D}_{(i+j,r+k)}^* \quad \text{and vice versa in } \theta_{2p/q}(x).

(xxvii) On the similar lines as in (xii), we can find \theta_{3p/q} by replacing \mathcal{A}_{(i+j,r+k)} = \mathcal{B}_{(i+j,r+k)}^* = \mathcal{D}_{(i+j,r+k)}.

\mathcal{A}_{(i+j,r+k)}^* = \mathcal{B}_{(i+j,r+k)}^*, \quad \mathcal{C}_{(i+j,r+k)}^* = \mathcal{D}_{(i+j,r+k)}^* \quad \text{and vice versa in } \theta_{3p/q}(x).
(xxviii) On the similar lines as in (xxiv), we can find \( \theta_{4p^q} (k) \) by replacing \( C^{(i+j,r+k)} = D^{(i+j,r+k)} \) and Viceversa in \( \theta_{4p^q} (k) \).

(xxix) On the similar lines as in (xxv), we can find \( \theta_{3p^q} (k) \) by replacing \( D_{(i+j,r+k)} = X_{(i+j,r+k)} \) and Viceversa in \( \theta_{3p^q} (k) \).

(30x) On the similar lines as in (xxvi), we can find \( \theta_{2p^q} (k) \) by replacing \( B_{(i+j,r+k)} = E_{(i+j,r+k)} \) and Viceversa in \( \theta_{2p^q} (k) \).

(31x) On the similar lines as in (xxvii), we can find \( \theta_{2p^q} (k) \) by replacing \( D_{(i+j,r+k)} = X_{(i+j,r+k)} \) and Viceversa in \( \theta_{2p^q} (k) \).

(32x) On the similar lines as in (xxviii), we can find \( \theta_{2p^q} (k) \) by replacing \( D_{(i+j,r+k)} = X_{(i+j,r+k)} \) and Viceversa in \( \theta_{2p^q} (k) \).

(33x) On the similar lines as in (xxix), we can find \( \theta_{2p^q} (k) \) by replacing \( A_{(i+j,r+k)} = Y_{(i+j,r+k)} \), \( D_{(i+j,r+k)} = X_{(i+j,r+k)} \) and Viceversa in \( \theta_{2p^q} (k) \).

(34x) On the similar lines as in (xxxiv), we can find \( \theta_{p^q} (k) \) by replacing \( A_{(i+j,r+k)} = Y_{(i+j,r+k)} \), \( B_{(i+j,r+k)} = E_{(i+j,r+k)} \) and Viceversa in \( \theta_{p^q} (k) \).

where \( A_{(n-1,m-1)} = p^{n-1}q^{m-1}(\frac{-1+\gamma}{4}) \), \( B_{(n-1,m-1)} = p^{n-1}q^{m-1}(\frac{-1-\gamma}{4}) \), \( C_{(n-1,m-1)} = p^{n-1}q^{m-1}(\frac{-1+\gamma+\delta}{4}) \), \( D_{(n-1,m-1)} = p^{n-1}q^{m-1}(\frac{-1-\gamma+\delta}{4}) \), \( A^*_{(n-1,m-1)} = p^{n-1}q^{m-1}(\frac{1+\gamma}{4}) \), \( B^*_{(n-1,m-1)} = p^{n-1}q^{m-1}(\frac{1-\gamma}{4}) \), \( C^*_{(n-1,m-1)} = p^{n-1}q^{m-1}(\frac{1-\gamma+\delta}{4}) \), \( D^*_{(n-1,m-1)} = p^{n-1}q^{m-1}(\frac{1+\gamma+\delta}{4}) \).
\[
\eta_0 = \sum_{s=0}^{2^{n-1}-1} (\alpha^{2p^n-q^m})^s, \quad \eta_1 = \sum_{s=0}^{2^{n-1}-1} (\alpha^{2p^n-q^m})^{2s}, \quad \xi_0 = \sum_{s=0}^{2^{n-1}-1} (\alpha^{2p^n-q^m})^{2s}, \quad \xi_1 = \sum_{s=0}^{2^{n-1}-1} (\alpha^{2p^n-q^m})^{4s}.
\]

5. References

[6] Ranjeet Singh, “Some results in the Ring \( \mathbb{Z}/(p^nq^m-1) \) \]