Some Vertex Prime Graphs And A New Type Of Graph Labeling.
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Abstract

In this paper we investigate some new families of vertex prime graphs. We show that the graphs kayak paddle KP(k,m,t), book graph $\theta(C_m)^n$, irregular book graph $\theta(C)^n$, $C_3$ snake $S(C_3, m)$, $m$-fold triangular snake $S(C_3, m, n)$, sunflower graph $SF(1, n)$, $m$-fold-petal sunflower graph $SF(m, n), (C_n)^k$ and *One point union of $k$ cycles not of equal length $(C)^k$ has Vertex Prime label. The * indicates new families are discussed and shown to be vertex prime. We introduce a new type of graph labeling called as L-cordial labeling and show that $K_{1, n}$, path $P_n$, $C_n$, $S(C_3, m)$ are families of L-cordial graphs.

Key words: Vertex Prime labeling, L-cordial labeling, path, cycle.

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1. Introduction

All graphs considered here are simple, finite, connected, undirected. A graph $G(V, E)$ has vertex prime labeling if it’s edges can be labeled with distinct integers $1, 2, 3, \ldots |E|$ i.e. a function $f: E \rightarrow \{1, 2, \ldots |E|\}$ defined such that for each vertex with degree at least 2 the greatest common divisor of the labels on its incident edges is 1 [4]. The edge labels are the actual images under function $f: E(G) \rightarrow \{1, 2, \ldots |E|\}$. Deretsky, Lee and Mitchen [4] shows that the forests, all connected graphs, $C_2UC_n, 5C_{2m}$ have vertex prime labeling. The graph with exactly two components one of which is not an odd cycle has a vertex prime labeling. They also show that a 2-regular graph with at least two odd cycles has no vertex prime labeling. The graph admitting vertex prime labeling is called as vertex prime graph.

We introduce a new type of graph labeling called as L-cordial labeling and show that $K_{1, n}$, path $P_n$, $C_n$, $S(C_3, n)$ are families of L-cordial graphs.

We refer for terminology and symbols J.F.Harary [7] and Dynamic survey of graph labeling by Galian J.A. [5]

In this paper main results follow after some definitions.

1.1 Definition: A book graph $\theta(C_m)^n$ is made from $m$ copies of cycle $C_m$ that share an edge in common. It is an $n$ page book with each page as a $m$-gon.

1.2 Definition: Irregular book graph $\theta(C)^n$ is a book on $n$ pages such that not all cycles are identical polygons.

1.3 Definition: A kayak paddle $G = KP(k, m, t)$ is a graph obtained by joining cycle $C_k$ and cycle $C_m$ by a path of length $t$.

1.4 Definition: A graph $S(C_3, m)$ is a snake of length $m$ on $C_3$. It is obtained from a path $p_m=(v_1, v_2, \ldots, v_{m+1})$ by joining vertices $v_i$ and $v_{i+1}$ to new vertex $w_i$ (i.e. $1, 2, \ldots, m$) giving edges $q_i=(w_i, v_i)$ and edge $q_i'=(w_i, v_{i+1})$. 

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1.5 Definition: A m-fold triangular snake \( S(C_3,m,n) \) of length \( n \) is obtained from a path \( v_1,v_2,\ldots,v_n,v_{n+1} \) by joining \( v_i \) and \( v_{i+1} \) to new \( m \) vertices \( w_{ij} \) for \( i = 1,2,\ldots,n \) giving edges \( (v_iw_{ij}) \) and \( e'_{ij} = (w_{ij}v_{i+1}) \), \( i = 1,2,\ldots,n \).

1.6 Definition: A unifold-petal sunflower graph \( SF(1,n) \) is a sunflower graph \( SF(n) \) obtained by a \( n \)-cycle \( (v_1,v_2,\ldots,v_n,v_1) \) and creating new vertices \( w_1,w_2,\ldots,w_n \) with new edges \( (v_iw_i) \) and \( (w_iv_{i+1}) \). It can also be obtained from a \( C_3 \) snake \( S(C_3,n) \) by identifying end points \( v_1 \) and \( v_{n+1} \) of path \( P_{n+1} \).

1.7 Definition: A m-fold-petal sunflower graph \( SF(m,n) \) is obtained from a cycle \( v_1,v_2,\ldots,v_n,v_1 \) by joining \( v_i \) and \( v_{i+1} \) to new \( m \) vertices \( w_{ij} \) for \( i = 1,2,\ldots,n \) giving edges \( e_{ij} = (v_iw_{ij}) \) and \( e'_{ij} = (w_{ij}v_{i+1}) \), \( i = 1,2,\ldots,n \), taken modulo \( n \).

1.8 Definition: \( (C_n)^k \) is a graph obtained by taking one point union of \( k \) copies of cycle \( C_n \).

1.9 Definition: \( (C)^k = (C_{r_1r_2\ldots r_k})^k \) is a graph obtained by taking one point union of \( k \) cycles of lengths \( r_1, r_2, \ldots, r_k \) not all same.

2. Main Results proved:

2.1 Theorem: A book graph \( \Theta(C_m)^n \) is a vertex prime.

Proof: A book graph \( G = \Theta(C_m)^n \) has \( n(m-1)+1 \) edges. The \( n \) copies of cycle \( C_m \) are denoted by \( C_1,C_2,\ldots,C_n \). The consecutive edges on cycle \( C_k \) are \( e_{i1} = (v_{i1}v_{i2}) \). (it is common to all cycles and is denoted by \( e_1 \)) \( , e_{21}, e_{k1} \). Where \( k = 1,2,\ldots,n \) and vertex \( v_{ij} \) is the \( j \)-th vertex on \( i \)-th cycle. The edge \( e_{ij} = (v_{ij}v_{i+1}) \) is the \( j \)-th edge on \( i \)-th cycle, \( i = 1,2,\ldots,n \) and \( j = 1,2,\ldots,m \). The edge \( e_1 \) is adjacent to \( e_{k1} \) and \( e_{k2} \) for \( k = 1,2,\ldots,n \). Define \( f : E(G) \rightarrow \{1,2,\ldots|E|\} \) such that

\[
\begin{align*}
  f(e_{i1}) &= x, x = 1,2,\ldots,m. \\
  f(e_{i2}) &= m+x-1, x = 2,3,\ldots,m. \\
  f(e_{ij}) &= m+(m-1)(k-2)+x-1; k=3,4,\ldots,n \text{ and } x = 2,3,\ldots,m. 
\end{align*}
\]

fig.2.1 \( \Theta(C_6)^2 \)

For any vertex the incident edges have label numbers that are consecutive positive integers or one of the incident edge is of label 1. Thus graph \( G \) is vertex prime.

2.2 Theorem: Irregular book graph \( G = \Theta(C)^p \) or \( G = \Theta(C_{r_1,r_2,\ldots r_k})^n \) has vertex prime labeling.
Proof: The different n cycles are $C_{r_1}, C_{r_2}, \ldots, C_{r_n}$. These cycles are also referred as $1^{\text{st}}, 2^{\text{nd}}, \ldots, n^{\text{th}}$ cycle on edges $r_1, r_2, \ldots, r_n$ respectively. The edge common to all cycles is $e_1$. $e^k_x$ be the $x^{\text{th}}$ edge on $k^{\text{th}}$ cycle where $x = 2, 3, \ldots, r_k$\ and $k = 1, 2, 3, \ldots, n$.

Define $f: E(G) \to \{1, 2, \ldots, |E|\}$ as

$$f(e^k_x) = x \quad \text{for } k=1 \text{ and } x=1, 2, \ldots, r_1$$
$$= \sum_{i=1}^{k-1} r_i - k + 1 + x \quad \text{for } k=2, 3, \ldots, n \text{ and } x=2, 3, \ldots, r_k.$$

It follows that for any vertex on any cycle $C_{r_1}, C_{r_2}, \ldots, C_{r_n}$ the incident edges have label numbers that are consecutive positive integers or one of the incident edge is of label 1. As such the graph $\theta(C)^n = \Theta(C_{r_1}, C_{r_2}, \ldots, C_{r_n})$ has vertex prime labeling.

2.3 Theorem: A kayak paddle $G= KP(k,m,t)$ is vertex prime.

Proof: A Kayak graph $G= KP(k,m,t)$ has vertex set $V(G)= \{v_1, v_2, \ldots, v_k, u_1, u_2, \ldots, u_{t-1}, w_1, w_2, \ldots, w_m\}$ where the cycle $C_k = (v_1, v_2, \ldots, v_k, v_1)$ and path of length $t = P_{t+1} = (v_1, u_1, u_2, \ldots, u_{t-1}, w_1)$ and $C_m = (w_1, w_2, \ldots, w_m, w_1)$. The edge set $E(G) = \{e_i = (v_i v_{i+1}), i = 1, 2, \ldots, k\}$ where $k+1$ is taken as 1} $\cup \{p_i = (v_i u_i), p_t = (u_t w_1) \cup \{p_{i+1} = (u_i u_{i+1}), i=1, 2, \ldots, t-2. \} \cup \{e'_i = (w_i w_{i+1}), i=1, 2, \ldots, m, v_{m+1} \text{ is taken as } v_1\}$

Define a function $f: E(G) \to \{1, 2, \ldots, |E|\}$ as

$$f(e_i) = i \text{ for } i = 1, k, f(p_i) = k+1, f(p_t) = k+1; \quad i = 1, 2, \ldots, t-1.$$
$$f(p_t) = k+t;$$
$$f(e'_i) = k+t+i; \quad i = 1, m.$$ This gives every vertex is incident with at least two edges whose labels are consecutive positive integers. Therefore G is vertex prime.

2.4 Theorem: A $C_3$ snake $G = S(C_3, m)$ is vertex prime.

Proof: Define $f: E(G) \to \{1, 2, \ldots, |E|\}$ as

$$f(e_i) = i \text{ for } e_i = (v_i v_{i+1}) \quad i = 1, 2, \ldots, m$$
$$f(e'_i) = m+1+2(m-i), \quad i = 1, 2, \ldots, m,$$
$$f(e_i) = f(e'_i) + 1 \quad i = 1, 2, m$$
As each vertex is incident with edges whose label are consecutive integers, the resultant labeling is vertex prime.

2.5 Theorem: A sunflower graph \( SF(n) \) \( n \geq 3 \) has vertex prime labeling.

Proof. We first obtain the vertex prime labeling of \( S(C3,n) \) as stated in theorem 1.4. By identifying the vertex \( v_1 \) with vertex \( v_{n+1} \) will give the required labeling of \( SF(n) \).

2.6 Theorem: A \( m \)-fold triangular snake \( S(C_3,m,n) \) is vertex prime.

Proof: Define \( f:E(G) \rightarrow \{1,2,\ldots,|E|\} \) as follows,

\[
f(e_i) = i, \quad i = 1,2,\ldots,n \]
\[
f(e_i') = n + 2n(j-1) + 2(n-i) + 1, \quad j = 1,2,\ldots,m \]
\[
f(e_j') = f(e_j') + 1 \]

It follows that \( S(C_3,m,n) \) is vertex prime.

2.7 Theorem: A \( m \)-fold-petel sunflower graph \( SF(m,n) \) is vertex prime.

Proof: A \( m \)-fold-petel sunflower graph \( SF(m,n) \) is obtained from \( S(C_3,m,n) \) by identifying vertex \( v_1 \) and \( v_n \) of \( S(C_3,m,n) \). Identifying these two vertices do not have any effect on vertex prime labeling. The resultant graph is vertex prime.

2.8 Theorem: The graph \( G = (C_{r1r2\ldots rk})^k \) is vertex prime.
Proof: Let the k cycles be C_j, j=1,2..k. with length r_1,r_2,...r_k respectively. The vertex common to all cycles be v_1 and e_i=(v_i,v_{i+1}) i = 1,2..n.

Define f:E(G)→{1,2,...|E|} as follows,
\[ f(e_j) = r_1 + r_2 + ... + r_{(j-1)} + i \] for j = 1..k

This labeling produces at least two edges on any vertex with label as consecutive natural numbers. Resultant graph is vertex prime.

If we take r_1=r_2=r_3=...=r_k we get all cycles of same length say m. The resultant graph is \((C_m)^k\)

3. Future challenges in vertex prime labelings:

3.1 Definition A block - cutpoint graph of a graph G is a bipartite graph in which one partite set consists of the cut vertices of G and the other has a vertex b_i for each block B_i of G.

3.2 Definition A triangular cactus is a connected graph all of whose blocks are triangles. On similar lines one can define n-gonal cactus as a graph in which all of its blocks are n-gone.

We define n-gonal snake on the same lines as the triangular snakes.

3.3 We observe that n-gonal snake is vertex prime. Obtain the particular labeling to this effect.

3.4 A n-gonal cactus is vertex prime. Obtain the particular labeling to this effect.

4. We define a new type of labeling: L-cordial labeling of a graph.

A graph G(V,E) has a L-cordial labeling if there is a bijective function f:E(G)→{1,2,...|E|}. This induces the vertex label as 0 if among all the labels on the incident edges the biggest label is an even number and 1 otherwise. Further the condition is satisfied that \(v(0)\) the number of vertices labeled with 0 and \(v(1)\) the number of vertices labeled with 1 follows the condition that \(|v(1)-v(0)|\leq1\). Here isolated vertices are not considered for labeling. A graph which admits L-cordial labeling is called as L-cordial graph.

4.1 Theorem: \(K_{1,n}\) is L-cordial iff n is even.

Proof: Label the pendent edges in \(K_{1,n}\) as 1,2,3..n

Case : n = 2m m= 0,1,2... we have m edges with odd label and m edges with even label producing m vertices with label 0 and m vertices with label 1. The vertex with n degree will receive label 0. Thus \(|v(0) - v(1)|\leq1\).

Case : n = 2m + 1. (m = 0,1,2...) There will be m+1 edges with odd label number and m edges with even number as label. This will giving m+1 vertices with label 1 and m vertices with label 0. Since the biggest edge label at the n degree vertex is odd number the vertex label will be 1. The resultant labeling gives \(v(0) = m\) and \(v(1) = m+2\). which is not L-cordial labeling.

4.2 Theorem: A path \(G = P_n\) is L-cordial. n≥3

Proof: case n=2 there is no L-cordial labeling of \(P_2\).

Case n=3 \(v_1=1\) \(v_2=0\) \(v_3=0\)

\[
\begin{array}{c}
1 \\
2 \\
3
\end{array}
\]
case \( n = 2m + 1, m= 1,2,3\ldots \)

Define \( f:E(G)\rightarrow \{1,2,\ldots,|E|\} \) as follows:
\[
f(e_1)=1, f(e_2)=3, f(e_3)=2, f(e_i)=i \text{ for } i\geq 4, v(f(0))= m, v(f(1))=m+1
\]

Case \( n =2m, m = 1,2,\ldots \):

Define \( f:E(G)\rightarrow \{1,2,\ldots,|E|\} \) as follows:
\[
f(e_1)=2, f(e_2)=1, f(e_3)=3, f(e_i)=i \text{ for } i\geq 4, v(f(0))= m, v(f(1))=m
\]

The resultant labeling is L-cordial labeling.

4.3 Theorem: Cycle \( C_n \) is L-cordial.

Proof: Let the cycle be \( C_n=(v_1,v_2,\ldots,v_n,v_1) \) and any edge \( e_i=(v_iv_{i+1}), i= 1,2,\ldots,n \) (\( n+1 \) taken modulo \( n \))

case 1 : \( n \) is even.

Define \( f:E(G)\rightarrow \{1,2,\ldots,E\} \) as follows,
\[
f(e_1)=1, f(e_2)=3, f(e_3)=2, f(e_i)=i \text{ for } i = 4,5,\ldots,n.
\]

The edge with label number \( n \) is adjacent to the edge with label 1. \( v(f(0))= v(f(1))= n/2 \). For odd cycle \( ( n=2m+1) \) Redefine the function as \( f(e_i)=i \) for \( i = 1,2,3,\ldots,n. \) gives \( v(f(0)) + 1= v(f(1))\# \)

4.4 Theorem: \( S(C_3,m) \) is L-cordial.

Proof: Define \( f:E(G)\rightarrow \{1,2,\ldots,|E|\} \) as
\[
f(v_{i}v_{i+1})= 3(i-1)+1 \quad i=1,2,\ldots,m.
f(w_jv_j)= 2 +(j-1)3 \quad j=1,2,\ldots,m
\]
\[
f(w_jv_{j+1})= 3j \quad j=1,2,\ldots,m \quad \text{When } n \text{ is odd } v(f(1)) = v(f(0)) +1 \text{ and when } n \text{ is even } v(f(0)) = v(f(1)) +1. \text{ Thus } S(C_3,n) \text{ is L-cordial.}
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