Sum Perfect Square Graphs in context of some graph operations

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Abstract

A \((p, q)\) graph \(G = (V, E)\) is called sum perfect square if for a bijection \(f : V(G) \rightarrow \{0, 1, 2, \ldots, p - 1\}\) there exists an injection \(f^* : E(G) \rightarrow \mathbb{N}\) defined by \(f^*(uv) = (f(u))^2 + (f(v))^2 + 2f(u) \cdot f(v), \forall uv \in E(G)\). Here \(f\) is called sum perfect square labeling of \(G\). In this paper we prove that \(P_n^2, K_n^2, K_{1,n} \cup K_{1,n+1}, mC_n, mK_{1,n}, spl(K_{1,n})\) and \(D_2(K_{1,n})\) are sum perfect square graphs. Further we prove that the union of path graph with any sum perfect square graph is also sum perfect square graph.

Key words: Sum perfect square graph.

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1 Introduction

Sonchhatra and Ghodasara 4 initiated the study of sum perfect square graphs. Due to 4 it becomes possible to construct a graph, whose all edges can be labeled by different perfect square integers. In the same paper the authors proved that \(P_n, C_n, C_n\) with one chord, \(C_n\) with twin chords, \(T_{m,n}\), \(K_{1,n}\), \(T_{m,n}\) are sum perfect square graphs. Sonchhatra and Ghodasara 4 proved that several snakes related graphs are sum perfect square. The same authors found some new sum perfect square graphs in 6.

2 Literature survey and Previous work

Throughout this paper we consider graph \(G = (p, q)\) (with \(p\) vertices and \(q\) edges) to be simple, finite and undirected. The set of vertices and edges of \(G\) are denoted by \(V(G)\) and \(E(G)\) respectively. For all other terminology and notations we follow Harary 1.

Definition 2.1 (1). Let \(G = (p, q)\) be a graph. A bijection \(f : V(G) \rightarrow \{0, 1, 2, \ldots, p - 1\}\) is called sum perfect square labeling of \(G\), if the induced function \(f^* : E(G) \rightarrow \mathbb{N}\) defined by \(f^*(uv) = (f(u))^2 + (f(v))^2 + 2f(u) \cdot f(v)\) is injective, \(\forall uv \in E(G)\).

A graph which admits sum perfect square labeling is called sum perfect square graph.

Definition 2.2 (3). The square of a graph \(G\) is denoted by \(G^2\), where \(V(G^2) = V(G)\) and two vertices \(u, v \in V(G^2)\) are adjacent in \(G^2\) if and only if \(d(u, v) \leq 2\) in \(G\), where \(d(u, v)\) denotes the distance between \(u\) and \(v\).

Definition 2.3 (3). The union of two graphs \(G_1 = (V_1, E_1)\) and \(G_2 = (V_2, E_2)\) is denoted by \(G_1 \cup G_2\), where \(V(G_1 \cup G_2) = V_1 \cup V_2\) and \(E(G_1 \cup G_2) = E_1 \cup E_2\).

Definition 2.4 (3). For a graph \(G\) the splitting graph \(spl(G)\) is obtained from \(G\) by taking two copies of \(G\) say \(G'\) and \(G''\) adding for each vertex \(v'\) of \(G'\) a new vertex \(v''\) of \(G''\) so that \(v'\) is adjacent to every vertex that is adjacent to \(v''\).

Definition 2.5 (3). For a graph \(G\) the shadow graph is denoted by \(D_2(G)\) is obtained from \(G\) by...
adding a new vertex $v'$ corresponding to a vertex $v$ of $G$, so that $v'$ is adjacent to every vertex that is adjacent to $v$.

3 Main Results

**Theorem 3.1.** $P_n^2$ is sum perfect square graph, $\forall n \in \mathbb{N} - \{1\}$.

**Proof.** Let $V(P_n^2) = \{v_i; 1 \leq i \leq n\}$ and $E(P_n^2) = \{e_i = v_iv_{i+1}; 1 \leq i \leq n-1\} \cup \{e_0 = v_1v_{n+1}; 1 \leq i \leq n-2\}$.

$|V(P_n^2)| = n$ and $|E(P_n^2)| = 2n - 3$.

We define a bijection $f : V(P_n^2) \rightarrow \{0, 1, 2, \ldots, n - 1\}$ by

$f(v_i) = i - 1, 1 \leq i \leq n$.

Let $f^* : E(P_n^2) \rightarrow \mathbb{N}$ be the induced edge labeling function defined by $f^*(uv) = (f(u))^2 + (f(v))^2 + 2f(u) \cdot f(v)$, $\forall uv \in E(P_n^2)$.

**Injectivity for edge labels:**

For $1 \leq i \leq n - 1$, $f^*(e_i^{(1)})$ is increasing in terms of $i$.

$\Rightarrow f^*(v_iv_{i+1}) < f^*(v_{i+1}v_{i+2}), 1 \leq i \leq n - 3$.

Similarly $f^*(e_i^{(2)})$ is increasing for $1 \leq i \leq n - 2$.

Further $f^*(e_i^{(1)})$ are odd perfect square integers and $f^*(e_i^{(2)})$ are even perfect square integers.

So $f^* : E(P_n^2) \rightarrow \mathbb{N}$ is injective and hence $P_n^2$ is sum perfect square graph, $\forall n \in \mathbb{N} - 1$.

The below illustration provides better idea about the above defined labeling pattern.

![Figure 1: Sum perfect square labeling of $P_n^2$.](attachment:of_path.pdf)

**Theorem 3.2.** $K_n^2$ is sum perfect square graph, $\forall n \in \mathbb{N} - \{1\}, n < 4$.

**Proof.** We know that $K_n^2 \cong K_n$, and $K_n$ is sum perfect square graph, for $n < 4$ (See Theorem 3.5 in [4]). Hence $K_n^2$ is sum perfect square graph, $\forall n \in \mathbb{N} - 1, n < 4$.

**Theorem 3.3 ([4]).** $P_n$ is sum perfect square graph, $\forall n \in \mathbb{N}$.

**Corollary 3.4.** For any sum perfect square graph $G$, $G \cup P_n$ is also sum perfect square graph, $\forall n \in \mathbb{N}$.

**Proof.** Let $G$ be a sum perfect square graph of order $n$. Consider a sum perfect square labeling $g : V(G) \rightarrow \{0, 1, 2, \ldots, n - 1\}$. By shifting the range set of sum perfect square labeling defined for $P_n$ (See [2] lemma 3.1) to $\{n, n+1, \ldots, n+m-1\}$ and combination of above two labelings gives required sum perfect square labeling.

**Corollary 3.5.** $K_{1,m} \cup K_{1,n}$ is sum perfect square graph, $\forall m, n \in \mathbb{N}$.

**Proof.** Let $V(K_{1,m} \cup K_{1,n}) = \{u_i, v_i; 1 \leq i \leq m\} \cup \{v_i, v_i; 1 \leq i \leq n\}$ and $E(K_{1,m} \cup K_{1,n}) = \{uu_i; 1 \leq i \leq m\} \cup \{vv_i; 1 \leq i \leq n\}$.

$|V(K_{1,m} \cup K_{1,n})| = m+n+2$ and $|E(K_{1,m} \cup K_{1,n})| = m+n$.

We define a bijection $f : V(K_{1,m} \cup K_{1,n}) \rightarrow \{0, 1, 2, \ldots, m+n+1\}$ by

$f(u_i) = 0$.

$f(v_i) = i; 1 \leq i \leq m$.

$f(v_i) = m+i; 1 \leq i \leq n$.

Let $f^* : E(K_{1,m} \cup K_{1,n}) \rightarrow \mathbb{N}$ be the induced edge labeling function defined by $f^*(uv) = (f(u))^2 + (f(v))^2 + 2f(u) \cdot f(v), \forall uv \in E(K_{1,m} \cup K_{1,n})$.

**Injectivity for edge labels:**

As $f$ is strictly increasing (for increasing values of $i$) we get $\{f^*(uv_i); 1 \leq i \leq m\}$ and $\{f^*(vv_i); 1 \leq i \leq n\}$ are distinct.

Also max $\{f^*(uv_i)\} = m^2 > \min\{f^*(vv_i)\} = (2m+n+2)^2$. Therefore $f^*$ is injective and hence $K_{1,m} \cup K_{1,n}$ is sum perfect square graph, $\forall m, n \in \mathbb{N}$.

However the authors strongly believe that the union of any two sum perfect square graphs is also a sum perfect square graph. So authors put the following conjecture.

**Conjecture 3.6.** For any two sum perfect square graphs $G$ and $H$, $G \cup H$ is sum perfect square graph.

**Theorem 3.7.** $mK_{1,n}$ is sum perfect square graph, $\forall m, n \in \mathbb{N}$.

**Proof.** Let $V(mK_{1,n}) = \{u_i; 1 \leq i \leq m\} \cup \{v_i; 1 \leq i \leq m, 1 \leq j \leq n\}$, where $\{v_i; 1 \leq i \leq m\}$ is the apex vertex of $i^{th}$ copy of $K_{1,n}$ and $E(mK_{1,n}) = \{u_iv_{i+1}; 1 \leq i \leq m, 1 \leq j \leq n\}$.

$|V(mK_{1,n})| = mn$ and $|E(mK_{1,n})| = mn$. We define a bijection $f : V(mK_{1,n}) \rightarrow \{0, 1, 2, \ldots , mn+1\}$ by
Injectivity for edge labels:

As $f$ is increasing (for increasing values of $i$ and $j$) all $\{f^*(u_{ij})\}$ will be smaller than the smallest edge label of $(f_{t+1}(i_{t+1}j))$, $1 \leq t \leq m-1$, $1 \leq j \leq n$. Hence the induced edge labeling $f^*: E(mK_{1,n}) \rightarrow \mathbb{N}$ is injective and so $mK_{1,n}$ is sum perfect square graph, $\forall n \in \mathbb{N}$. □

**Theorem 3.8.** $mC_n$ is sum perfect square graph, $\forall m,n \in \mathbb{N}, n \geq 3$.

**Proof.** Let $V(mC_n) = \{v_i; 1 \leq i \leq n, 1 \leq j \leq m\}$, and $E(mC_n) = \{u_{ij}v_{i+1}; 1 \leq i \leq n, 1 \leq j \leq m\}$. 
$|V(mC_n)| = mn$, $|E(mC_n)| = mn$. 
We define a bijection $f : V(mC_n) \rightarrow \{0,1,2,\ldots,mn-1\}$ by

$$f(v_i) = \begin{cases} 2i - 2 + (j - 1)n; 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil, \\ 2(n - i) + 1 + (j - 1)n; \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n. 
\end{cases}$$

and $1 \leq j \leq m$.

Injectivity for edge labels:

For any one copy of $mC_n$, it follows from theorem 3.2 that induced edge labels are injective. Moreover the largest edge label of $C_n$ is smaller than the smallest edge label of $(t+1)C_n$, $1 \leq t \leq m-1$. Hence the induced edge labeling $f^*: E(mC_n) \rightarrow \mathbb{N}$ is injective.

Therefore $mC_n$ is sum perfect square graph, $\forall m,n \in \mathbb{N}, n \geq 3$. □

**Theorem 3.9.** $spl(K_{1,n})$ is sum perfect square graph, $\forall n \in \mathbb{N}$.

**Proof.** Let $V(spl(K_{1,n})) = \{u, u_i; 1 \leq i \leq n\} \cup \{v, v_i; 1 \leq i \leq n\}$ and $E(spl(K_{1,n})) = \{u_iu; 1 \leq i \leq n\} \cup \{v_iv; 1 \leq i \leq n\} \cup \{uv_i; 1 \leq i \leq n\} \cup \{uv; 1 \leq i \leq n\} \cup \{vu_i; 1 \leq i \leq n\}$.
$|V(spl(K_{1,n}))| = 2n + 2$, $|E(spl(K_{1,n}))| = 4n$. We define a bijection $f : V(spl(K_{1,n})) \rightarrow \{0,1,\ldots,2n+1\}$ by

$$f(u) = 0.$$ $f(u_i) = i; 1 \leq i \leq n.$
$f(v) = 2n + 1.$
$f(v_i) = n + i; 1 \leq i \leq n.$

Let $f^*: E(spl(K_{1,n})) \rightarrow \mathbb{N}$ be the induced edge labeling function defined by $f^*(uv) = (f(u)^2 + (f(v))^2) + 2(f(u) \cdot f(v)), \forall uv \in E(spl(K_{1,n})).$

Injectivity for edge labels:

We note that $f^*(uu_i)$ is increasing for increasing values of $i, 1 \leq i \leq n$ and hence all $f^*(uu_i)$ are distinct. Similarly $f^*(v_iu), f^*(vu_i)$ and $f^*(vu_i)$ are also distinct.

Moreover

$$(1) \quad \text{max}\{f^*(uu_i)\} \leq \text{min}\{f^*(v_iu), f^*(vu_i), f^*(vu_i)\}.$$ $$(2) \quad \text{min}\{f^*(v_iu)\} \geq \text{max}\{f^*(vu_i), f^*(vu_i), f^*(vu_i)\}.$$ $$(3) \quad \text{max}\{f^*(vu_i)\} < \text{min}\{f^*(vu_i)\}.$$ $f^*: E(spl(K_{1,n})) \rightarrow \mathbb{N}$ is injective and hence $spl(K_{1,n})$ is sum perfect square graph, $\forall n \in \mathbb{N}$. □

The below illustration provides better idea about the above defined labeling pattern.

![Figure 3: Sum perfect square labeling of spl(K_{1,3})](http://www.ijmttjournal.org)
Figure 4: Sum perfect square labeling of $D_2(K_{1,4})$.

4 Conclusion

In this paper several sum perfect square graphs have been found in context of the graph operations, which are union of graphs, splitting of a graph and shadow graph. A conjecture has been posed related to union of any two sum perfect square graphs.

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References


