Nash Equilibrium Solution of Trapezoidal Fuzzy Number in Bi-matrix Game

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Abstract: Bi-matrix game with symmetric trapezoidal pay-off is considered in this paper. At first trapezoidal number ranking method for such games is defined and then we define Nash equilibrium solution for pure strategies and mixed strategies. The inequality constraints involving trapezoidal coefficient are reduced in their satisfactory crisp equivalent form and a satisfactory solution of the problem is established. Numerical example is given to illustrate the methodology.

Keywords: Symmetric trapezoidal number, Bi-Matrix game Nash equilibrium.

I. INTRODUCTION

Game theory is a mathematical tool using which conflicting interest situations is handled. In recent times much attention has been drawn to interval valued game, Nayak and Pal [1, 2, 3], Narayanan [4], Nishizaki [5]. In practical situations the pay-offs are given within certain ranges rather than as an exact number. These uncertain situations are overcome when we use interval numbers as pay-offs. An interval number is an extension of a real number and also when we use interval numbers as pay-offs, these uncertain situations are overcome.

In dominance method [1], if the convex combination of any two rows (columns) of a pay-off matrix is dominated by the third row (column) then it indicates that the third move of the row (column) of the player will be an optimal move but we are not certain as to which one of the first two moves will be an optimal one. This disadvantage is overcome through the graphical method [2]. However, this is not the method of solution but a state of art technique, where an mx2 or 2xn particular type interval game is reduced to 2x2 interval sub games. But the situation may arise where more than two decision makers are involved and that situation is handled by bi-matrix game. Fuzziness, in bi-matrix games, was studied by many authors [8, 9, 10]. Nayak and Pal [11] described bi-matrix games with interval pay-off and its Nash equilibrium strategy. In this paper we have described Bi-matrix game with trapezoidal fuzzy pay off. We have described the arithmetic operation and inequality relation of trapezoidal fuzzy numbers. We defined Nash equilibrium solution of such games and tried to get the solution as a trapezoidal fuzzy number. The numerical example illustrates the theory.

II. SYMMETRIC TRAPEZOIDAL NUMBER

Here we will pay attention on some definitions and notations of symmetric trapezoidal fuzzy number. Our aim is to define a fuzzy symmetric trapezoidal matrix in space of matrices. The triangular symmetric fuzzy number has already been defined. Here we define the symmetric trapezoidal fuzzy number over the real line L in some different manner as

\[ L = \{ a \in \mathbb{R} \ \text{and} \ \alpha = 2a_o + (4x-1)a, \ x \in [0, 1] \} \]

Here at

\[ x = 0, a = 2a_o - \alpha \]

\[ x = \frac{1}{4}, a = 2a_o \]

\[ x = \frac{1}{2}, a = 3a_o + \alpha \]

Where \( a_o \) is the arithmetic mean value \( (\text{for} \ x = \frac{1}{4}) \), is the 2nd arithmetic mean value \( \text{for} \ x = \frac{1}{2}, \ q \) is the lower bound and \( a \) is the upper bound. The symmetric trapezoidal membership function is defined on [0, 1] as

\[ \Pi (x) = \begin{cases} 0 & \text{when} \ 0 \leq x < \frac{1}{4} \\ 4x & \text{when} \ 0 < x < \frac{1}{2} \\ 1 & \text{when} \ \frac{1}{4} \leq x \leq \frac{3}{4} \\ 4(1-x) & \text{when} \ \frac{3}{4} < x < 1 \\ 0 & \text{when} \ x \geq 1 \end{cases} \]

Here one observation can be made that we use three parameters to represent a symmetric trapezoidal fuzzy number \((m, m, \omega)\) though four parameters are required to represent a general trapezoidal fuzzy number. This representation has been made for the simplicity of the computation.
III. ARITHMETIC OF TRAPEZOIDAL FUZZY NUMBER

An extension of ordinary arithmetic to trapezoidal fuzzy number is known as trapezoidal arithmetic. Let $A=(a_1,a_2,\omega)$ and $B=(b_1,b_2,\omega)$ be two trapezoidal numbers. The arithmetic and multiplication by a real number ‘c’ are defined as follows:

1. The sum of two trapezoidal numbers is a trapezoidal number i.e
   \[A+B=(a_1,a_2,\omega)+(b_1,b_2,\omega)\]
   \[= (a_1+b_1,a_2+b_2,\omega+\omega)\]

2. The subtraction of two trapezoidal fuzzy number is also a trapezoidal fuzzy number i.e
   \[A-B=(a_1,a_2,\omega)-(b_1,b_2,\omega)\]
   \[= (a_1-b_2,a_2-b_1,\omega+\omega)\]

3. If $c \neq 0$ be a scalar then $cA=(ca_1,ca_2,\omega)$, if $c \geq 0$ and $cA=(ca_1,ca_2,-\omega)$ if $c < 0$

4. The product of two trapezoidal number is given by
   \[AB=\min\{a_1b_1-\frac{1}{2}(a_1b_2+a_2b_1)+\frac{1}{4}\omega,\]
   \[a_1b_1-\frac{1}{2}(a_1b_2-a_2b_1)-\frac{1}{4}\omega,\]
   \[a_2b_1+\frac{1}{2}(a_2b_2+a_2b_1)+\frac{1}{4}\omega,\]
   \[a_2b_1+\frac{1}{2}(a_2b_2-a_2b_1)-\frac{1}{4}\omega\}\]
   \[\max\{a_1b_1-\frac{1}{2}(a_1b_2+a_2b_1)+\frac{1}{4}\omega,\]
   \[a_1b_1-\frac{1}{2}(a_1b_2-a_2b_1)-\frac{1}{4}\omega,\]
   \[a_2b_1+\frac{1}{2}(a_2b_2+a_2b_1)+\frac{1}{4}\omega,\]
   \[a_2b_1+\frac{1}{2}(a_2b_2-a_2b_1)-\frac{1}{4}\omega\}\]
   \[(max-min)\]

5. The division of these two interval numbers A and B ($\neq 0$) is
   \[\frac{A}{B} = \min\{\frac{2a_1}{2b_1}-\frac{1}{2},\frac{2a_1}{2b_1}+\frac{1}{2}\},\]
   \[\max\{\frac{2a_1}{2b_1}-\frac{1}{2},\frac{2a_1}{2b_1}+\frac{1}{2}\}\]
   \[(max-min)\]

Provided $2b_1 \neq w, 2b_2 + w \neq 0$.

6. $A \cdot A \neq 0$
7. $\frac{A}{A} \neq 1$

Example: If $A=(3,7,4)$ and $B=(4,12,8)$ are two trapezoidal numbers then $A+B=(7,19,12)$. A-B = (-9,3,12), AB = (0,144,144), $\frac{A}{B}$ does not exist as $2b_1 = w$.

IV. INEQUALITY RELATION WITH TRAPEZOIDAL NUMBERS

A comparison between trapezoidal numbers is given in this section.

Case 1: Let $A=(a_L,a_R,\omega)$ and $B=(b_L,b_R,\omega)$ be two trapezoidal numbers in $T(\mathbb{R})$, the set of trapezoidal numbers. Here we make analogous order relation to Moore [6] as $A \leq B$ (A is strictly less than B) i.e if $\omega + \omega' < 2(b_L - a_R)$. Here this relation is an extension of ‘<’ on real line. This relation seems to be strict order relation that A is smaller than B.

Example: Let $A=(5,6,1)$ and $B=(6,14,8)$ be two trapezoidal numbers. Here $\omega + \omega' = 9$ and $2(b_L - a_R) = 2(14 - 6) = 16$ implies $A < B$.

Case 2: Let $A=(a_L,a_R,\omega)$ and $B=(b_L,b_R,\omega)$ $\in T(\mathbb{R})$ such that

\[2a_L - \omega \leq 2b_L - \omega < 2b_R + \omega \leq 2a_R + \omega \]

Then $B$ contained in $A$, denoted by $B \subseteq A$ which is the extension of the concept of the set inclusion. The ranking order of A and B can be defined as

\[A \oplus B = \begin{cases} \{B, \text{ if the player is pessimistic} \} & \text{if } A \leq B \\ \{A, \text{ if the player is optimistic} \} & \text{if } A < B \end{cases}\]

Where the notation $A \oplus B$ represents maximum among two trapezoidal and

\[A \triangle B = \begin{cases} \{B, \text{ if the player is optimistic} \} & \text{if } A \leq B \\ \{A, \text{ if the player is pessimistic} \} & \text{if } A < B \end{cases}\]

Where the notation $A \triangle B$ represents minimum among two trapezoidal.

Example: Let $A=(3,9,6)$ and $B=(4,8,4)$ are two trapezoidal s.t. $B \subseteq A$ for optimistic player $AVB = A$ and $A \triangle B = B$. For pessimistic player $AVB = B$ and $A \oplus B = A$.

Case 3: Let us consider the case where the trapezoidal A and B are not contained in others their some portions are common. In that case we define acceptability index to compare and order two trapezoidal numbers.

4.1 Acceptability Index

We use acceptability index $AI$ to compare and order any two trapezoidal numbers on the real line in terms of value as in [12, 2], which are used throughout the paper.

Let us consider two symmetric trapezoidal $(m_1,m_1,a_1)$ and $(m_2,m_2,\omega_2)$.

**Definition** For $m_1 + m_2 \leq m_2 + m_1$ and $\omega_1 + \omega_2 \neq 0$ the value judgment index or acceptability index (AI) of the premise $A < B$ is defined by

\[A=\frac{m_1 + m_2}{\omega_1 + \omega_2}\]

\[B=\frac{m_2 + m_1}{\omega_1 + \omega_2}\]

\[\frac{A}{B} \text{ is defined by ...} \]
\[ AI (A' < B') = \frac{(m_1 + m_1') - (m_2 + m_2')}{2(\omega_1 + \omega_2)} \] 

which is the value judgment by which \( A' \) is inferior to \( B' \) (\( B' \) is superior to \( A' \)) in terms of value. Here ‘inferior to’, ‘superior to’, are analogous to ‘less than’, ‘greater than’, respectively.

The above observations can be put into a compact form as follows:

\[
A' \lor B' =
\begin{cases} 
B', & \text{if } AI(A' < B') > 0 \\
A', & \text{if } AI(A' < B') = 0 \text{ and } \omega_1 < \omega_2 \text{ DM is pessimistic.} \\
B', & \text{if } AI(A' < B') = 0 \text{ and } \omega_1 > \omega_2 \text{ DM is optimistic.}
\end{cases}
\]

Similarly, in the following we have given another function \( max' \) which determines the maximum between two trapezoidal numbers.

Function \( max' (A', B') \)

If \( A = B \) then maximum = \( A' \); 
else 
if\( A' = (m_1, m_1', \omega_1) \) and \( B' = (m_2, m_2', \omega_2) \) and are not non-dominating then
maximum = \( B' \); 
else
maximum = \( A' \); 
endif; 
else if\( \omega_1 > \omega_2 \) then
if the decision maker is pessimistic maximum = \( A' \); 
else if the decision maker is optimistic maximum = \( B' \); 
endif; 
endif; 
return(maximum); 
End Function.

Similarly, if \( m_1 + m_1' \geq m_2 + m_2' \) and \( \omega_1 \geq \omega_2 \) and, then there also exist a strict preference relation between \( A' \) and \( B' \). Thus similar observations can be put into a compact form as

\[
A' \land B' =
\begin{cases} 
B', & \text{if } AI(B' < A') > 0 \\
A', & \text{if } AI(B' < A') = 0 \text{ and } \omega_1 > \omega_2 \text{ DM is pessimistic.} \\
B', & \text{if } AI(B' < A') = 0 \text{ and } \omega_1 > \omega_2 \text{ DM is optimistic.}
\end{cases}
\]

V. INEQUALITY CONSTRAINTS INVOLVING TRAPEZOIDAL COEFFICIENTS

In this section we concentrate on the LPP in which both the objective function and coefficients of constraints are all trapezoidal fuzzy numbers. Linear programming problems with interval coefficients have been studied by several researchers [3] by introducing some preference relations between interval numbers. We will follow analogous method to get inequality constraints involving trapezoidal coefficients. Following the definition of AI the inequality constraints involving trapezoidal coefficients are reduced in their crisp equivalent forms so that they can be solved easily.

Let \( A = (a_{ij}, a_{Ri}, \omega_i) \) and \( B = (b_{ij}, b_{Ri}, \omega_i) \in \mathbb{T}(\mathbb{R}) \) and \( x \) is a singleton variable. Then acceptability condition of the inequality constraints \( Ax \leq B \) may be defined as

\[ AI (B' < A') \geq 0 \]

\[
\Rightarrow m(Ax) + m'(Ax) \leq m(B) + m'(B)
\]

Hence a crisp equivalent form of trapezoidal equality relation may be defined as

\[ Ax \leq B \]

\[
\Rightarrow \begin{cases} 
(2a_1 + \omega) x \leq (2b_R + \omega)
\end{cases}
\]

Where \( \leq \) denotes trapezoidal number inequality and \( \alpha \) denotes the minimal acceptance degree of constraints fixed by decision maker. Similarly the inequality constraints \( Ax \leq B \), the satisfactory crisp equivalent can be defined in the form

\[ Ax \geq B \]

\[
\Rightarrow \begin{cases} 
(2a_1 - \omega) x \geq (2b_R - \omega)
\end{cases}
\]

Let \( c_{ij}, c_{Ri}, \omega_{cj} \), \( a_{ij}, a_{Ri}, \omega_{aj} \), \( b_{lj}, b_{Rj}, \omega_{bj} \) \( \in \mathbb{T}(\mathbb{R}) \). We consider following trapezoidal number linear programming problem (TNLPP) as

\[
\max f(x) = \sum (c_{ij}, c_{Ri}, \omega_{cj}) x_i 
\]

s.t \( \sum (a_{ij}, a_{Ri}, \omega_{aj}) x_i \leq (b_{lj}, b_{Rj}, \omega_{bj}) \)

When max denotes fuzzy maximization, the crisp number \( x_i \) are unknown, \( f(x) \) is the objective function, 2nd relation denotes inequality constraints ‘m’ denotes the number of constraints and ‘n’ denotes the number of variables. This type of model is called TNLPP. However classical LPP technique cannot be applied to get solution unless the above trapezoidal valued structure be reduced into standard LPP structure. In particular when the parameters are degenerated into crisp real numbers, TNLPP turns into classical LPP as
max \ Z = m(f(x)) = \frac{1}{2}\{\sum[C_{ij} + C_{ji}]x_j\} = \frac{1}{2}[V_L + V_R] \\
 ...(10)

s.t.\ \Sigma(2a_{ij} + \omega_{ai})x_i \geq 2V_L, i = 1, 2, \ldots, m \ ...(11)

\Sigma[(1 - \alpha)(2a_{ij} + \omega_{ai})]x_i \geq 1 + \alpha V_L + 1 - \alpha V_R \quad \ldots \quad ...(12)

\Sigma x_i = 1, \ x_i \geq 0, j = 1, 2, 3, \ldots, n; \ \alpha \in [0, 1]

The solution of this type crisp optimization problem satisfies all constraints exactly. In the analogous trapezoidal valued problem the degree of satisfaction of objective(s) and constraints is minimized.

VI. TRAPEZOIDAL VALUED BI-MATRIX GAME

The TVBG is a finite non cooperative two person game. It can be considered as a natural extension of classical game to cover situations in which outcome of a decision process does not necessarily dictate the verdict that what one player gains and other has to lose. The table showing how payments should be made at the end of the game is called a payoff matrix. Let 1 and 2 denote two decision makers (DM) and let M = \{1, 2, 3, \ldots, m\} and N = \{1, 2, 3, \ldots, n\} be the sets of all pure strategies available for DM1 and DM2 respectively. A Nash equilibrium solution represents an equilibrium point where each player reacts to other by choosing the option that gives him/her preference.

7.1 Pure Strategy

Let 1 and 2 denote two decision makers(DM) and let M=\{1,2,3,4…m\} and \ N=\{1,2,3,4 …n\} be the sets of all pure strategies available for DM1 and DM2 respectively. A Nash equilibrium solution of the TVBG is said to constitute a Nash equilibrium solution for a TVBG \( \Gamma \) if the following pair of inequalities are satisfied for all \ i = 1, 2, \ldots, m and \ j = 1, 2, \ldots, n.

\[ a_{ij} \geq \alpha \quad \text{and} \quad \beta_{ij} \geq \beta \]

The pair \( (\alpha_{ij}, \beta_{ij}, \omega_{ij}) \) is said to be a Nash equilibrium outcome of the TVBG.

7.3 Mixed strategy

A probabilistic situation arises when a player does not know the decision the other player takes but guesses what he can do and in that case the objective function is to maximize the expected gain. Such strategies are known as mixed strategies. Let \( \Re_m \) and \( \Re_n \) be the \( m \) and \( n \) dimensional nonnegative Euclidean spaces, respectively. Denote \( x = (x_1, x_2, x_3, \ldots, x_m) \) and \( y = (y_1, y_2, y_3, \ldots, y_n) \) respectively, where the symbol ‘\( T \)’ denotes the transpose of a vector. The strategy spaces for players and are denoted as,

\[ S_A = \{(x_1, x_2, \ldots, x_m) \in \Re_m^+; i = 1, 2, \ldots, m \quad \text{and} \quad \sum_{i=1}^{m} x_i = 1\} \]

\[ S_B = \{(y_1, y_2, \ldots, y_n) \in \Re_n^+; i = 1, 2, \ldots, n \quad \text{and} \quad \sum_{i=1}^{n} y_i = 1\} \]

respectively. Vectors \( x \in S_A \) ,\( y \in S_B \) are called mixed strategies of players A and B respectively.

Definition 2 (Trapezoidal expected pay-off): If the mixed strategies \( x = (x_1, x_2, x_3, \ldots, x_m) \) and \( y = (y_1, y_2, y_3, \ldots, y_n) \) are proposed by players A and B respectively, then the expected pay-off of the player by player A and B is defined by

\[ x^T y = \sum_{i=1}^{m} \sum_{j=1}^{n} x_i (\alpha_{ij}, \beta_{ij}, \omega_{ij}) y_j \quad \text{and} \quad d^T y = \sum_{i=1}^{n} \sum_{j=1}^{m} y_j (\gamma_{ij}, \delta_{ij}, \omega_{ij}) x_i \]

Where we assume that each of the two players chooses a strategy, a payoff for each of them and it is represented as trapezoidal number. We call this game a TVBG and denote the game by \( \Gamma = \{(1, 2), \ A, \ B\} \).
Addition and other composition rules on trapezoidal numbers (discussed in section 2.2) are used in this definition (14) of expected pay-offs.

**Definition 3 Nash equilibrium solution**
A pair \(\{x^*, y^*\}\) is called a Nash equilibrium solution to a TVBG in mixed strategies if the following inequalities are satisfied.

\[
x^T A y^* \leq x^T A y^*; \forall x \in S_1; \quad x^T B y^* \leq x^T B y^*; \forall y \in S_2
\]

\(x^*\) and \(y^*\) are also called the optimal strategies for A and B respectively. Then the pair of intervals \(V = (x^T A y^*, x^T B y^*)\) is known as the Nash equilibrium outcome of the TVBG \(\Gamma\) in mixed strategies, and the triplet \((x^*, y^*, V)\) is said to be a solution of TVBG.

**VIII. NUMERICAL EXAMPLE**

Consider a 2x2 TVBG with the pay-off matrices as

\[
A = \begin{bmatrix}
\begin{array}{cc}
\frac{2}{3} & \frac{4}{3} \\
\frac{5}{3} & \frac{7}{3}
\end{array}
\end{bmatrix}, B_1 = \begin{bmatrix}
(1,1,0)
\end{bmatrix}, B_2 = \begin{bmatrix}
\frac{1}{3}, \frac{1}{3}, \frac{1}{3}
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
\begin{array}{cc}
\frac{6}{7} & \frac{10}{7} \\
\frac{5}{7} & \frac{3}{7}
\end{array}
\end{bmatrix}, A_1 = \begin{bmatrix}
\frac{1}{3}, \frac{1}{3}, \frac{1}{3}
\end{bmatrix}, A_2 = \begin{bmatrix}
\frac{2}{3}, \frac{2}{3}, \frac{2}{3}
\end{bmatrix}
\]

Here, Nash equilibrium solution in pure strategies does not exist w.r.t the ranking order described in section 4. Here every \(x \in S_1\) can be written as \((x_1, 1-x_1)^T\) with \(0 \leq x_1 \leq 1\) and similarly every \(y \in S_2\) as \((y_1, 1-y_1)\) with \(0 \leq y_1 \leq 1\) we get the solution of the game as

\[x^* = \left(\frac{1}{2}, \frac{1}{2}\right)^T\] and

\[y^* = \left(\frac{1}{2}, \frac{1}{2}\right)\] and \(V = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)\)

**IX. CONCLUSION**

In this paper, we have considered a bi-matrix game whose pay-off elements are symmetric trapezoidal fuzzy numbers. Arithmetic operations and inequality relations of the symmetric trapezoidal fuzzy numbers are described and Nash equilibrium strategies are explained. The numerical example establishes the theory on strong ground. It has wide industrial application where conflicting interest situations exist.

**REFERENCES**


