ON SOFT $\delta$-CONTINUOUS FUNCTIONS

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Abstract. In this paper, we introduce a new class of functions called soft $\delta$-continuous functions. We obtain several characterizations and some of their properties. Also, we investigate its relationship with other types of functions.

1. Introduction

The concept of soft sets was first introduced by Molodtsov [11] in 1999 as a general mathematical tool for dealing with uncertain objects. In [11, 12], Molodtsov successfully applied the soft theory in several directions, such as smoothness of functions, game theory, operations research, Riemann integration, Perron integration, probability, theory of measurement, and so on.

After presentation of the operations of soft sets [10], the properties and applications of soft set theory have been studied increasingly [3, 7, 12]. In recent years, many interesting applications of soft set theory have been expanded by embedding the ideas of fuzzy sets [1, 2, 4, 8, 9, 10, 12]. To develop soft set theory, the operations of the soft sets are redefined and a uni-int decision making method was constructed by using these new operations [5].
Recently, in 2011, Shabir and Naz [14] initiated the study of soft topological spaces. They defined soft topology on the collection $\tau$ of soft sets over $X$. Consequently, they defined basic notions of soft topological spaces such as soft open and soft closed sets, soft subspace, soft interior, soft closure, soft neighborhood of a point, soft separation axioms, soft regular spaces and soft normal spaces and established their several properties. Hussain and Ahmad [6] investigated the properties of soft open, soft closed, soft interior, soft closure, soft neighborhood of a point. They also defined and discussed the properties of soft interior, soft exterior and soft boundary which are fundamental for further research on soft topology and will strengthen the foundations of the theory of soft topological spaces.

In this paper, we introduce a new class of functions called soft $\delta$-continuous functions. We obtain several characterizations and some of their properties. Also, we investigate its relationship with other types of functions.

2. Preliminaries

In this section, we present some basic definitions and results which are needed in further study of this paper which may found in earlier studies. Throughout this paper, $X$ refers to an initial universe, $E$ is a set of parameters, $\mathcal{P}(X)$ is the power set of $X$, and $A \subseteq E$

Definition 2.1. [11] A soft set $F_A$ over the universe $X$ is defined by the set of ordered pairs

$F_A = \{(e, F_A(e)) : e \in E, F_A(e) \in \mathcal{P}(X)\}$

where $F_A : E \rightarrow \mathcal{P}(X)$, such that $F_A(e) \neq \emptyset$, if $e \in A \subseteq E$ and $F_A(e) = \emptyset$ if $e \notin A$. The family of all soft sets over $X$ is denoted by $SS(X)$.

Definition 2.2. [10] The soft set $F_\emptyset$ over a common universe set $X$ is said to be null soft set, denoted by $\emptyset$. Here $F_\emptyset(e) = \emptyset$, $\forall e \in E$.

Definition 2.3. [10] A soft set $F_A$ over $X$ is called an absolute soft set, denoted by $\tilde{A}$, if $e \in A$, $F_A(e) = X$.

Definition 2.4. [10] Let $F_A, G_B$ be soft sets over a common universe set $X$. Then $F_A$ is a soft subset of $G_B$, denoted $F_A \subseteq G_B$ if $F_A(e) \subseteq G_B(e)$, $\forall e \in E$.

Definition 2.5. [10] Let $F_A, G_B$ be soft sets over a common universe set $X$. The union of $F_A$ and $G_B$, is a soft set $H_C$ defined by $H_C(e) = F_A(e) \cup G_B(e)$, $\forall e \in E$, where $C = A \cup B$.

That is, $H_C = F_A \cup G_B$.

Definition 2.6. [10] Let $F_A, G_B$ be soft sets over a common universe set $X$. The intersection of $F_A$ and $G_B$, is a soft set $H_C$ defined by $H_C(e) = F_A(e) \cap G_B(e)$, $\forall e \in E$, where $C = A \cap B$. 

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That is, \( H_C = F_A \cap G_B \).

**Definition 2.7.** [14] The complement of the soft set \( F_A \) over \( X \), denoted by \( F_A^c \), is defined by \( F_A^c(e) = X - F_A(e) \), \( \forall e \in E \).

**Definition 2.8.** [14] Let \( F_A \) be a soft set over \( X \) and \( x \in X \). We say that \( x \in F_A \) if \( x \in F_A(e) \), \( \forall e \in A \). For any \( x \in X \), \( x \notin F_A \) if \( x \notin F_A(e) \) for some \( e \in A \).

**Definition 2.9.** [16] The soft set \( F_A \in SS(X) \) is called a soft point in \( SS(X) \) if there exist \( x \in X \) and \( e \in E \) such that \( F(e) = \{ x \} \) and \( F(e^c) = \emptyset \) for each \( e^c \in E - \{ e \} \) and the soft point \( F_A \) is denoted by \( x \in \).

**Definition 2.10.** [14] A soft topology \( \tau \) is a family of soft sets over \( X \) satisfying the following properties.

1. \( \emptyset, X \) belong to \( \tau \).
2. The union of any number of soft sets in \( \tau \) belongs to \( \tau \).
3. The intersection of any two soft sets in \( \tau \) belongs to \( \tau \).

The triplet \( (X, \tau, E) \) is called a soft topological space.

**Definition 2.11.** [13] Let \( (X, \tau, E) \) be a soft topological space over \( X \). Then

1. The members of \( \tau \) are called soft open sets in \( X \).
2. A soft set \( F_A \) over \( X \) is said to be a soft closed set in \( X \) if \( F_A^c \in \tau \).
3. A soft set \( F_A \) is said to be a soft neighborhood of a point \( x \in X \) if \( x \in F_A \) and \( F_A \) is soft open in \( (X, \tau, E) \).
4. The soft interior of a soft set \( F_A \) is the union of all soft open subsets of \( F_A \). The soft interior of \( F_A \) is denoted by \( \text{int}(F_A) \).
5. The soft closure of \( F_A \) is the intersection of all soft closed super sets of \( F_A \). The soft closure of \( F_A \) is denoted by \( \text{cl}(F_A) \) or \( F_A \).

**Definition 2.12.** [15] A soft set \( F_A \) in a soft topological space \( (X, \tau, E) \) is said to be a soft regular open (resp. soft regular closed) if \( F_A = \text{int}(\text{cl}(F_A)) \) (resp. \( F_A = \text{cl}(\text{int}(F_A)) \)).

### 3. Soft \( \delta \)-open sets

**Definition 3.1.** Let \( F_A \) be a soft subset of soft topological space \( (X, \tau, E) \). Then

1. \( x_e \) is called a soft \( \delta \)-cluster point of \( F_A \) if \( F_A \cap \text{int}(\text{cl}(U_A)) \neq \emptyset \) for every soft open set \( U_A \) containing \( x_e \).
2. The family of all soft \( \delta \)-cluster point of \( F_A \) is called the soft \( \delta \)-closure of \( F_A \) and is denoted by \( \text{cl}_\delta(F_A) \).
Lemma 3.2. Let $F_A$ be a soft subset of soft topological space $(X, \tau, E)$. Then, the following properties hold:

1. $\text{int}(\text{cl}(F_A))$ is soft regular open,
2. Every soft regular open set is soft $\delta$-open,
3. Every soft $\delta$-open set is the union of a family of soft regular open sets.
4. Every soft $\delta$-open set is soft open.

Proof. (1) Let $F_A$ be a soft subset of $X$ and $G_A = \text{int}(\text{cl}(F_A))$. Then, we have $\text{int}(\text{cl}(G_A)) = \text{int}(\text{cl}(F_A)) = G_A$. Therefore $G_A$ is soft regular open.

(2) Let $F_A$ be a soft regular open. For each $x_\varepsilon \in F_A$, $(X-F_A) \cap F_A = \emptyset$ and $F_A$ is soft regular open. Hence $x_\varepsilon \notin \text{cl}_S(X-F_A)$ for each $x_\varepsilon \in F_A$. This shows that $x_\varepsilon \notin (X-F_A)$ implies $x_\varepsilon \notin \text{cl}_S(X-F_A)$. Therefore, we have $\text{cl}_S(X-F_A) \subset X-F_A$ which implies $\text{cl}_S(X-F_A) = X-F_A$ and hence $F_A$ is soft $\delta$-open.

(3) Let $F_A$ be a soft $\delta$-open set. Then $X-F_A$ is soft $\delta$-closed and hence $X-F_A = \text{cl}_S(X-F_A)$. For each $x_\varepsilon \in F_A$, $x_\varepsilon \notin \text{cl}_S(X-F_A)$ and there exists a soft open neighborhood $O_{x_\varepsilon}$ such that $\text{int}(\text{cl}(O_{x_\varepsilon})) \cap (X-F_A) = \emptyset$. Therefore, we have $x_\varepsilon \in O_{x_\varepsilon} \subset \text{int}(\text{cl}(O_{x_\varepsilon})) \subset F_A$ and hence $F_A = \cup \{\text{int}(\text{cl}(O_{x_\varepsilon})) : x_\varepsilon \in F_A\}$. By (1), $\text{int}(\text{cl}(O_{x_\varepsilon}))$ is soft regular open for each $x_\varepsilon \in F_A$.

(4) This follows from Definition 3.1.

Proposition 3.3. Intersection of two soft regular open sets is soft regular open.

Proof. Let $F_A$ and $G_A$ be soft regular open. Then, we have $\text{int}(F_A \cap G_A) = \text{int}(F_A) \cap \text{int}(G_A) = \text{int}(F_A) \cap \text{int}(G_A) = \text{int}(F_A \cap G_A)$.

Lemma 3.4. Let $F_A$ and $G_A$ be soft subsets of soft topological space $(X, \tau, E)$. Then, the following properties hold:

1. $F_A \subset \text{cl}_S(F_A)$,
2. If $F_A \subset G_A$, then $\text{cl}_S(F_A) \subset \text{cl}_S(G_A)$,
3. $\text{cl}_S(F_A) = \cap \{G_A \in SS(X) : F_A \subset G_A\}$ and $G_A$ is soft $\delta$-closed,
4. If $(F_A)_\alpha$ is a soft $\delta$-closed set of $X$ for each $\alpha \in \Delta$, then $\cap \{(F_A)_\alpha : \alpha \in \Delta\}$ is soft $\delta$-closed,
5. $\text{cl}_S(F_A)$ is soft $\delta$-closed.

Proof. (1) For any $x_\varepsilon \in F_A$ and any soft open neighborhood $U_A$ of $x_\varepsilon$, we have $\emptyset \neq F_A \cap U_A \subset F_A \cap \text{int}(U_A)$ and hence $x_\varepsilon \in \text{cl}_S(F_A)$. 

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(2) Suppose that $x \notin \text{cl}_\delta(G A)$. There exists a soft open neighborhood $U_A$ of $x$ such that
\[
\emptyset = \text{int}(\text{cl}(U_A)) \cap G A \quad \text{and hence} \quad \text{int}(\text{cl}(U_A)) \cap F_A = \emptyset.
\]
Therefore, we have $x \notin \text{cl}_\delta(F A)$.

(3) Suppose that $x \in \text{cl}_\delta(F A)$. For any soft open neighborhood $U_A$ of $x$, there exists a soft $\delta$-closed set $G A$ containing $F A$, which we have $\emptyset \notin F A \cup \text{int}(\text{cl}(U_A)) \subseteq G A \cap \text{int}(\text{cl}(U_A))$ and hence $x \in \text{cl}_\delta(G A) = G A$. This shows that $x \in \{G A \subseteq \text{SS}(X) : F A \subseteq G A \text{ and } G A \text{ is soft } \delta\text{-closed}\}$. Conversely, suppose that $x \notin \text{cl}_\delta(F A)$, then there exists a soft open neighborhood $U_A$ of $x$ such that $\text{int}(\text{cl}(U_A)) \cap F_A = \emptyset$. By Lemma 3.2, $X - \text{int}(\text{cl}(U_A))$ is a soft $\delta$-closed set which contains $F A$ and does not contain $x$. Therefore, $x \notin \bigcap \{G A \subseteq \text{SS}(X) : F A \subseteq G A \text{ and } G A \text{ is soft } \delta\text{-closed}\}$.

(4) For each $\alpha \in \Delta$, $\text{cl}_\delta(\cap_{\alpha \in \Delta}(F A)_{\alpha}) \subseteq \text{cl}_\delta((F A)_{\alpha}) = (F A)_{\alpha}$ and hence $\text{cl}_\delta(\cap_{\alpha \in \Delta}(F A)_{\alpha}) \subseteq \cap_{\alpha \in \Delta}(F A)_{\alpha}$. By (1), we obtain $\text{cl}_\delta(\cap_{\alpha \in \Delta}(F A)_{\alpha}) = \cap_{\alpha \in \Delta}(F A)_{\alpha}$. This shows that $\cap_{\alpha \in \Delta}(F A)_{\alpha}$ is soft $\delta$-closed.

(5) This follows immediately from (3) and (4).

**Theorem 3.5.** Let $(X, \tau, E)$ be a soft topological space and $\tau_\delta = \{F A \subseteq \text{SS}(X) : F A$ is a soft $\delta$-open set$\}$. Then $\tau_\delta$ is a soft topology weaker than $\tau$.

**Proof.** (1) It is obvious that $\emptyset, X \in \tau_\delta$.

(2) Let $(H_A)_{\alpha} \in \tau_\delta$ for each $\alpha \in \Delta$. Then $(H_A)_{\alpha}^c$ is soft $\delta$-closed for each $\alpha \in \Delta$. By Lemma 3.4, $\cap_{\alpha \in \Delta}(H_A)_{\alpha}^c$ is soft $\delta$ closed and $\cap_{\alpha \in \Delta}(H_A)_{\alpha} = (\cup_{\alpha \in \Delta}(H_A)_{\alpha})^c$. Hence $\cup_{\alpha \in \Delta}(H_A)_{\alpha}$ is soft $\delta$-open.

(3) Let $F A, G A \in \tau_\delta$. By Lemma 3.2, $F A = \cup_{\alpha_1 \in \Delta_1}(F A)_{\alpha_1}$ and $G A = \cup_{\alpha_2 \in \Delta_2}(G A)_{\alpha_2}$, where $(F A)_{\alpha_1}$ and $(G A)_{\alpha_2}$ are soft regular open sets for each $\alpha_1 \in \Delta_1$ and $\alpha_2 \in \Delta_2$. Thus $F A \cap G A = \bigcup \{(F A)_{\alpha_1} \cap (G A)_{\alpha_2} : \alpha_1 \in \Delta_1, \alpha_2 \in \Delta_2\}$. Therefore, $F A \cap G A$ is a soft regular open. Hence $F A \cap G A$ is a soft $\delta$-open set by Lemma 3.2.

4. soft $\delta$-continuous functions

**Definition 4.1.** Let $(X, \tau, E)$ and $(Y, \sigma, E)$ be two soft topological spaces and a mapping $f : (X, \tau, E) \rightarrow (Y, \sigma, E)$ is said to be soft $\delta$-continuous if for each $x \in \text{SS}(X)$ and each soft open neighborhood $V_A$ of $f(x)$, there exists a soft open neighborhood $U_A$ of $x$ such that $f(\text{int}(\text{cl}(U_A))) \subseteq \text{int}(\text{cl}(V_A))$.

**Theorem 4.2.** For a function $f : (X, \tau, E) \rightarrow (Y, \sigma, E)$, then the following properties are equivalent.

(1) $f$ is soft $\delta$-continuous.

(2) For each $x \in \text{SS}(X)$ and each soft regular open set $V_A$ containing $f(x)$, there exists a soft regular open set $U_A$ containing $x$ such that $f(U_A) \subseteq V_A$.

(3) $f(\text{cl}_\delta(F A)) \subseteq \text{cl}_\delta(f(F A))$ for every $F A \in \text{SS}(X)$.

(4) $\text{cl}_\delta(f^{-1}(K_A)) \subseteq f^{-1}(\text{cl}_\delta(K_A))$ for every $K_A \in \text{SS}(Y)$.
(5) For every soft δ-closed set $F_A$ of $Y$, $f^{-1}(F_A)$ is soft δ-closed in $X$,
(6) For every soft δ-open set $W_A$ of $Y$, $f^{-1}(W_A)$ is soft δ-open in $X$,
(7) For every soft regular open set $W_A$ of $Y$, $f^{-1}(W_A)$ is soft δ-open in $X$,
(8) For every soft regular closed set $G_A$ of $Y$, $f^{-1}(G_A)$ is soft δ-closed in $X$.

Proof. (1)⇒(2) This follows immediately from Definition 4.1.

(2)⇒(3) Let $x_e \in SS(X)$ and $F_A \in SS(X)$ such that $f(x_e) \in f(\overline{\delta}(F_A))$. Suppose that $f(x_e) \notin \overline{\delta}(f(F_A))$. Then, there exists a soft regular open neighborhood $H_A$ of $f(x_e)$ such that $f(F_A) \cap H_A = \emptyset$. By (2), there exists a soft regular open neighborhood $U_A$ of $x_e$ such that $f(U_A) \subseteq V_A$. Since $f(F_A) \cap f(U_A) \subseteq f(F_A) \cap V_A = \emptyset$, $f(F_A) \cap f(U_A) = \emptyset$. Hence, we get that $U_A \cap F_A \subseteq f^{-1}(f(U_A)) = f^{-1}(f(f(F_A))) = \emptyset$. Hence we have $U_A \cap F_A = \emptyset$ and $x_e \notin \overline{\delta}(F_A)$. This shows that $f(x_e) \notin \overline{\delta}(f(F_A))$. This is a contradiction. Therefore, $f(x_e) \in \overline{\delta}(f(F_A))$.

(3)⇒(4) Let $K_A \in SS(Y)$ such that $F_A = f^{-1}(K_A)$. By (3), $f(\overline{\delta}(f^{-1}(K_A))) \subseteq f(\overline{\delta}(f(f^{-1}(K_A))))$. Therefore, we have $\overline{\delta}(f^{-1}(K_A)) \subseteq \overline{\delta}(f(\overline{\delta}(f^{-1}(K_A)))) \subseteq \overline{\delta}(f^{-1}(\overline{\delta}(K_A)))$. Thus we obtain that $\overline{\delta}(f^{-1}(K_A)) \subseteq \overline{\delta}(f^{-1}(\overline{\delta}(K_A)))$.

(4)⇒(5) Let $F_A$ be soft δ-closed set of $Y$. By (4), $\overline{\delta}(f^{-1}(F_A)) \subseteq \overline{\delta}(\overline{\delta}(f^{-1}(F_A))) = \overline{\delta}(f(F_A))$ and always $f^{-1}(F_A) \subseteq \overline{\delta}(f^{-1}(F_A))$. Hence we obtain that $\overline{\delta}(f^{-1}(F_A)) = \overline{\delta}(f(F_A))$. This shows that $f^{-1}(F_A)$ is soft δ-closed.

(5)⇒(6) Let $W_A$ be soft δ-open set of $Y$. Then $\overline{\delta} - W_A$ is soft δ-closed. By (5), $f^{-1}(\overline{\delta} - W_A) = \overline{\delta} - f^{-1}(W_A)$ is soft δ-closed. Therefore, $f^{-1}(W_A)$ is soft δ-open.

(6)⇒(7) Let $W_A$ be soft regular open set of $Y$. Since every soft regular open set is soft δ-open, $W_A$ is soft δ-open. By (6), $f^{-1}(W_A)$ is soft δ-open.

(7)⇒(8) Let $F_A$ be soft regular closed set of $Y$. Then $\overline{\delta} - F_A$ is soft regular open. By (7), $f^{-1}(\overline{\delta} - F_A) = \overline{\delta} - f^{-1}(F_A)$ is soft δ-open. Therefore, $f^{-1}(F_A)$ is soft δ-closed.

(8)⇒(1) Let $x_e \in SS(X)$ and $V_A$ be soft open set containing $f(x_e)$. Now, set $H_A = \text{int}(\overline{\delta}(V_A))$, then by Lemma 3.2, $\overline{\delta} - H_A$ is a soft regular closed set. By (8), $f^{-1}(\overline{\delta} - H_A) = \overline{\delta} - f^{-1}(H_A)$ is a soft δ-closed set. Thus we have $f^{-1}(H_A)$ is soft δ-open. Since $x_e \in f^{-1}(H_A)$, by Lemma 3.2, there exists a soft open neighborhood $U_A$ of $x_e$ such that $x_e \in U_A \subseteq \text{int}(\overline{\delta}(V_A)) \subseteq f^{-1}(H_A)$. Hence $f(\text{int}(\overline{\delta}(U_A))) \subseteq \text{int}(\overline{\delta}(V_A))$. This shows that $f$ is soft δ-continuous function.

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