Two Kinds of Weakly Berwald Special ($\alpha, \beta$) - metrics of Scalar flag curvature

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Abstract: In this paper, we study two important class of special ($\alpha, \beta$)-metrics of scalar flag curvature in the form of $F = (\alpha + \beta)^2$ and $F = \sqrt{c_0 \alpha^2 + c_2 \alpha \beta + c_3 \beta^2}$ (where $c_1, c_2$ and $c_3$ are constants) are of scalar flag curvature. We prove that these metrics are weak Berwald if and only if they are Berwald and their flag curvature vanishes. Further, we show that the metrics are locally Minkowskian.

Key Words: Finsler metric, ($\alpha, \beta$) -metrics, S-curvature, Weak berwald metric, flag curvature, locally minkowski metric.

1. Introduction

The flag curvature in Finsler geometry is a natural extension of the sectional curvature in Riemannian geometry, which is first introduced by L. Berwald. For a Finsler manifold $(M, F)$, the flag curvature is a function $K(\alpha, \beta)$ of tangent planes $P \subset T_x M$ and directions $v \in P$. $F$ is said to be of scalar flag curvature if the flag curvature $K(\alpha, \beta) = K(x, y)$ is independent of flags $P$ associated with any fixed flagpole $y$. One of the fundamental problems in Riemann-Finsler geometry is to study and characterize Finsler metrics of scalar flag curvature. It is known that every Berwald metric is a Landsberg metric and also, for any Berwald metric the $S$-curvature vanishes [2]. In 2003, Shen proved that Randers metrics with vanishing $S$-curvature and $K = 0$ are not Berwaldian [10]. He also proved that the Bishop-Gromov volume comparison holds for Finsler manifolds with vanishing $S$-curvature.

The concept of ($\alpha, \beta$) -metrics and its curvature properties have been studied by various authors [1],[8],[4],[7]. X. Cheng, X. Mo and Z. Shen (2003) have obtained the results on the flag curvature of Finsler metrics of scalar curvature [3]. Yoshikawa, Okubo and M. Matsumoto(2004) showed the conditions for some ($\alpha, \beta$) -metrics to be weakly-Berwald. Z. Shen and Yildirim (2008) obtained the necessary and sufficient conditions for the metric $F = \frac{(\alpha + \beta)^2}{\alpha}$ to be projectively flat. They also obtained the necessary and sufficient conditions for the metric $F = \frac{(\alpha + \beta)^2}{\alpha}$ to be projectively flat Finsler metric of constant flag curvature and proved that, in this case, the flag curvature vanishes Recently, X. Cheng(2010) has worked on ($\alpha, \beta$) -metrics of scalar flag curvature with constant S-curvature. The main purpose of the present paper is to study and characterize the two important class of weakly-Berwald ($\alpha, \beta$) -metrics $F = \frac{(\alpha + \beta)^2}{\alpha}$ and $F = \sqrt{c_0 \alpha^2 + c_2 \alpha \beta + c_3 \beta^2}$ (where $c_1, c_2$ and $c_3$ are constants) are of scalar flag curvature The terminologies and notations are referred to [5][2].

2. Preliminaries

Let $M$ be an $n$-dimensional $C^\infty$ manifold and $TM = \cup_{x \in M} TxM$ denote the tangent bundle of $M$.

A Finsler metric on $M$ is a functions $F : TM \to [0, \infty)$ with the following properties:

a) $F$ is $C^\infty$ on $TM \setminus \{0\}$;

b) At each point $x \in M$, $F_x(y) = F(x, y)$ is a Minkowskian norm on $T_x M$.

The pair $(M, F)$ is called Finsler manifold;

Let $(M, F)$ be a Finsler manifold and

$$g_{ij}(x, y) = \frac{1}{2} [F^2(x, y)]_y A_{ij}$$

(2.1)

For a vector $y = y^i \frac{\partial}{\partial x^i} |_x \neq 0$, $F$ induces an inner product $g_y$ on $T_x M$ as follows $g_{ij}(u, v) = g_{ij}u^iv^j,$ where $u = u^i \frac{\partial}{\partial x^i} |_x \neq 0$ and $v = v^i \frac{\partial}{\partial x^i} |_x \neq 0$

Further, the Cartan torsion $C$ and the mean Cartan torsion $I$ are defined as follows[2]:
\[ C_y (x, y) = C_{ijk} u^i v^j w^k, \quad l_y (u) = l_i (x, y) u^i, \quad (2.2) \]
Where
\[ C_{ijk} (x, y) = \frac{1}{4} [F^2]_{ij} y^j y^k (x, y). \quad (2.3) \]
\[ l_i (x, y) = g^{jk} C_{ijk} (x, y) = \frac{\partial}{\partial y^j} \ln \sqrt{\det (g_{jk})}. \quad (2.4) \]
Where \((g^{jk}) = (g_{jk})^{-1}.\)

Let \(\alpha = \sqrt{a_{ij} y^i y^j}\) be a Riemannian metric and \(\beta = b_i (x) y^i\) be a 1-form on an \(n\)-dimensional manifold \(M.\) The norm
\[ \| \beta_x \|_x = \sqrt{a_{ij} b_i (x) b_j (x)}. \]
Let \(\varphi = \varphi (s)\) be a \(C^\infty\) positive function on an open interval \((-b_0, b_0)\) satisfying the following conditions
\[ \varphi (s) > 0, |s| \leq b < b_0. \]
Then the functions \(F = \alpha \varphi (\beta / \alpha)\) is a Finsler metric if and only if \(\| \beta_x \|_x = b_0.\) Such Finsler metrics are called \((\alpha, \beta)\) --metrics.

Let \(\gamma, \delta, \text{ and } \epsilon\) denote the horizontal covariant derivative with respect to \(F\) and \(\alpha,\) respectively.

Let
\[ r_i = \frac{1}{2} (b_{ij} + b_{ji}), \quad s_{ij} = \frac{1}{2} (b_{ij} - b_{ji}), \quad (2.5) \]
\[ s^i_j = a^{ik} s_{kj}, \quad s^i_i = b^i s^j_j = b^i s^j_i, \quad r_i = b^j r^j_i. \quad (2.6) \]
For a \(C^\infty\) positive function \(\varphi = \varphi (s)\) on \((-b_0, b_0)\) and a number \(b \in (0, b_0),\) let
\[ \Phi = -(Q - sQ') (n \Delta + 1 + sQ) (b^2 - s^2) (1 + sQ) \quad (2.7) \]
Where \(\Delta = 1 + sQ + b^2 - s^2 Q'\) and \(Q = \varphi' / (\varphi - s \varphi).\)

The geodesic \(x = x(t)\) of a Finsler metric \(F\) is characterized by the following system of second order ordinary differential equations:
\[ \frac{d^2 x^i (t)}{dt^2} + 2 G^i (x(t), x' (t)) = 0, \]
Where
\[ G^i = \frac{1}{4} g^{ij} \left\{ [F^2]_{xy} y^m F^2 (x) \right\}. \]
\(G^i\) is called as geodesic coefficients of \(F.\) For an \((\alpha, \beta)\) --metric \(F = \alpha \varphi (\beta / \alpha), s = \beta / \alpha,\) using a Maple program, we can get the following \([5]:\)
\[ G^i = \tilde{G}^i + \alpha Q s_0 + \Theta \left\{ -2 a Q s_0 + r_0 \right\} \left\{ \frac{y^j}{\alpha} + \frac{q}{q - s Q} b^j \right\}, \quad (2.8) \]
Where \(\tilde{G}^i\) denote the spray coefficients of \(\alpha.\)

We shall denote \(r_0 = r_{ij} y^i y^j, s_{10} = s_{ij} y^j, s_0 = s_i y^i\) and \(\Theta = \frac{q - s Q}{2 \Delta}.\)

Furthermore, let
\[ h_i = \alpha b_i - s y_i, \quad \Psi_1 = \frac{1}{\sqrt{b^2 - s^2 - 2 \Delta}} \frac{1}{\Delta^2} \left( \left[ \frac{b^2 - s^2}{3} \frac{\Phi}{\Delta^2} \right] \right) \quad (2.9) \]
By a direct computation, we can obtain a formula for the mean Cartan torsion of \((\alpha, \beta)\) --metrics as follows \([6]:\)
\[ l_t = \frac{\varphi (s - s \varphi)}{2 \Delta \varphi a^2} (\alpha b_i - s y_i). \quad (2.9) \]
According to Diecke’s theorem, a Finsler metric is Riemannian if and only if the mean Cartan torsion vanishes, \(l = 0.\) Clearly, an \((\alpha, \beta)\) --metric \(F = \alpha \varphi (\beta / \alpha), s = \beta / \alpha\) is Riemannian if and only if \(\Phi = 0\) (see \([6]):\)

For a Finsler metric \(F = F(x, y)\) on a manifold \(M,\) the Riemann curvature \(R_y = R^l_k \frac{\partial}{\partial x^l} \otimes dx^k\) is defined by
\[ R^l_k = \frac{\partial \sigma_i^l}{\partial x^k} - \frac{\partial \sigma_i^j}{\partial y^j} n_{\alpha \beta} + 2 \frac{\partial \sigma^l_i}{\partial y^m} \frac{\partial \sigma^m_j}{\partial y^k}, \]
Let \(R_{jk} = g_{ji} R^i_j,\) then \(R_{jk} y^j = 0, R_{jk} = R_{kj}.)
For a flag \( \{P, y\} \), where \( P = \text{span}\{y, u\} \subseteq TM \), the flag curvature \( K = K(P, y) \) of \( F \) is defined by

\[
K(P, y) = \frac{R_{i}(x, y)u^{i}u^{k}h_{ij}(x, y)u^{j}u^{k}}{F^{i}(x, y)h_{ij}(x, y)u^{i}u^{j}}. \tag{2.10}
\]

where \( h_{jk} = g_{jk} - F^{-2} - g_{j0}y^0g_{k0}y^0 \).

We say that Finlser metric \( F \) is of scalar flag curvature if the flag curvature \( K = K(x, y) \) is independent of the flag \( P \). By the definition, \( F \) is of scalar flag curvature \( K = K(x, y) \) if and only if in a standard local coordinate system,

\[
R_{i}^{l} = KF^{2}h_{i}^{l}. \tag{2.11}
\]

The Schur Lemma in Finsler geometry tells us that, in dimension \( n \geq 3 \), if \( F \) is of isotropic flag curvature, \( K = K(x) \), then it is constant flag curvature, \( K = \text{constant} \).

The Berwald curvature \( B_{p} = B_{ijkl}dx^{i} \otimes \partial / \partial x^{i} \otimes dx^{k} \otimes dx^{l} \) and mean Berwald curvature \( E_{p} = E_{ij}dx^{i} \otimes dx^{j} \) are defined respectively by

\[
B_{ijkl} = \frac{\partial^{3}G^{i}(x, y)}{\partial y^{j} \partial y^{k} \partial y^{l}}, \quad E_{ij} = \frac{1}{2} \frac{\partial^{2}}{\partial y^{j} \partial y^{l}} \left( \frac{\partial G^{m}}{\partial y^{m}} \right) = \frac{1}{2} B_{ijm}^{m}.
\]

A Finsler metric \( F \) is called weak Berwald metric if the mean Berwald curvature vanishes, i.e., \((E = 0)B = 0\). A Finsler metric \( F \) is said to be of isotropic mean Berwald curvature if \( E = \frac{1}{2} (n + 1)c(x)F^{-1}h \), where \( c = c(x) \) is a scalar function on the manifold \( M \).

**Theorem 2.1.** [9] For special \((\alpha, \beta)\) -metric \( F = (\alpha + \beta)^{2} \) and \( F = \sqrt{c_{1} \alpha^{2} + c_{2} \alpha \beta + c_{3} \beta^{2}} \) (where \( c_{1}, c_{2} \) and \( c_{3} \) are constants) an \( n \)-dimensional manifold \( M \). Then the following are equivalent:

(a) \( F \) is of isotropic \( S \)-curvature, \( S = (n + 1)c(x)F \);
(b) \( F \) is of isotropic mean Berwald curvature, \( E = \frac{n + 1}{2} c(x)F^{-1}h \);
(c) \( \beta \) is a killing 1-form with constant length with respect to \( \alpha \), i.e., \( \gamma_{00} = 0 \) and \( s_{0} = 0 \);
(d) \( S \)-curvature vanishes, \( S = 0 \);
(e) \( F \) is a weak Berwald metric, \( E = 0 \);

where \( c = c(x) \) is scalar function on the manifold \( M \).

Note that, the discussion in [9] doesn’t involve whether or not \( F \) is Berwald metric. By the definitions, Berwald metrics must be weak Berwald metrics but the converse is not true.

For this observation, we further study the metrics \( F = (\alpha + \beta)^{2} \) and \( F = \sqrt{c_{1} \alpha^{2} + c_{2} \alpha \beta + c_{3} \beta^{2}} \) (where \( c_{1}, c_{2} \) and \( c_{3} \) are constants).

For a Finsler metric \( F = F(x, y) \) on an \( n \)-dimensional manifold \( M \), the Busemann-Hausdorff volume form \( dV_{F} := \sigma_{F}(x)dx^{1} \wedge \ldots \wedge dx^{n} \) is given by

\[
\sigma_{F}(x) = \frac{\text{Vol}(R^{n+1})}{\text{Vol}(\{y \in R^{n} | F(x, y) < 1\})}.
\]

\( \text{Vol} \) denotes the euclidean volume in \( R^{n} \). The \( S \)-curvature is given by

\[
S(x, y) = \frac{\partial c^{m}}{\partial y^{m}} - \frac{c^{m}(n+\sigma_{F})}{\partial x^{m}}. \tag{2.12}
\]

Clearly, the mean Berwald curvature \( E_{ij} \) can be characterized by use of \( S \)-curvature as follows:

\[
E_{ij} = \frac{1}{2} \frac{\partial^{2}S}{\partial y^{i} \partial y^{j}}.
\]

A Finsler metric \( F \) is said to be of isotropic \( S \)-curvature if \( S = (n + 1)c(x)F \), where \( c = c(x) \) is a scalar function on the manifold \( M \). \( S \)-curvature is closely related to the flag curvature. Shen, Mo and cheng proved the following important result.

**Theorem 2.2.** [3] Let \((M, F)\) be an \( n \)-dimensional Finsler manifold of scalar flag curvature with flag curvature \( K = K(x, y) \). Suppose that the \( S \)-curvature is isotropic, \( S = (n + 1)c(x)F \), where \( c = c(x) \) is a scalar function on \( M \). Then there is a scalar function \( \sigma(x) \) on \( M \) such that
This shows that $S$-curvature has important influence on the geometric structures of Finsler metrics. For a Finsler metric $F$, the Landsberg curvature $L = L_{ijk} dx^i \otimes dx^j \otimes dx^k$ and the mean Landsberg curvature $J = J_k dx^k$ are defined respectively by

$$L_{ijk} = -\frac{1}{2} FF_{jkm}[G^m] y^i y^j y^k,$$
$$J_k = g^{ij} L_{ijk}.$$

A Finsler metric $F$ is called weak Landsberg metric if the mean Landsberg curvature vanishes, i.e., $J = 0$. For an $(\alpha, \beta)$--metric $F = \alpha \varphi(s), s = \beta/\alpha$, Li and Shen[8] obtained the following formula of the mean Landsberg curvature

$$J_i = -\frac{1}{2\alpha s^2} \left( \frac{\partial}{\partial s^2} \left[ \frac{\partial^2}{\partial \varphi^2} (G^i - \bar{G}^i) \right] (s_0 + r_0) h_i + \frac{\partial}{\partial s^2} \left[ \frac{\partial^2}{\partial \varphi^2} (G^i - \bar{G}^i) \right] r_{00} - 2\alpha Q s_0 h_i + \alpha Q (\alpha^2 s_i - y_i s_0 + \alpha^2 \Delta s_0) + [\alpha^2 (r_{00} - 2\alpha Q s_0) - r_{00} - 2\alpha Q s_0 y_i] s_0 \right).$$

Besides, they also obtained

$$J_i b^i = -\frac{1}{2\alpha s^2} \left( \frac{\partial}{\partial s^2} \left[ \frac{\partial^2}{\partial \varphi^2} (G^i - \bar{G}^i) \right] (s_0 + r_0) h_i + \frac{\partial}{\partial s^2} \left[ \frac{\partial^2}{\partial \varphi^2} (G^i - \bar{G}^i) \right] r_{00} - 2\alpha Q s_0 h_i + \alpha Q (\alpha^2 s_i - y_i s_0 + \alpha^2 \Delta s_0) + [\alpha^2 (r_{00} - 2\alpha Q s_0) - r_{00} - 2\alpha Q s_0 y_i] s_0 \right).$$

The horizontal covariant derivatives $J_i, m$ and $J_i b^i$ of $J_i$ with respect to $F$ and $\alpha$ respectively are given by

$$J_i m = \frac{\partial J_i}{\partial x^m} - J_i \Gamma^i_{jm} - \frac{\partial J_i}{\partial y^j} N^m_{jm} J_i m,$$
$$J_i b^i = \frac{\partial J_i}{\partial x^m} - J_i \Gamma^i_{jm} - \frac{\partial J_i}{\partial y^j} N^m_{jm} b^i.$$

Further we have,

$$J_{i,m} y^m = \left\{ J_{i,m} - J_i (\Gamma^i_{jm} - \Gamma^i_{jm}) - \frac{\partial J_i}{\partial y^j} (N^m_{jm} - \bar{N}^m_{jm}) \right\} y^m.$$

=\iota_{i,m} y^m - J_i (\Gamma^i_{jm} - \Gamma^i_{jm}) - \frac{\partial J_i}{\partial y^j} (N^m_{jm} - \bar{N}^m_{jm}) y^m.$$

If a Finsler metric $F$ is of constant flag curvature $K$ (see[13]), then

$$J_{i,m} y^m + K F^2 I_i = 0.$$

So, if an $(\alpha, \beta)$--metric $F = \alpha \varphi(s), s = \beta/\alpha$, is of constant flag curvature $K$, then

$$J_{i,m} y^m - J_i \frac{\partial (G^i - \bar{G}^i)}{\partial y^j} - \frac{\partial J_i}{\partial y^j} (G^i - \bar{G}^i) + K \alpha^2 \varphi^2 I_i = 0.$$

Contracting the above equation by $b^i$ yields the following equation

$$J_{i,m} b^i - J_i \frac{\partial (G^i - \bar{G}^i)}{\partial y^j} b^i - \frac{\partial J_i}{\partial y^j} (G^i - \bar{G}^i) + K \alpha^2 \varphi^2 I_i b^i = 0.$$ (2.16)

3. Characterization of Weakly-Berwald $(\alpha, \beta)$--metrics of Scalar flag curvature

In $n$-dimensional Finsler manifold ($n \geq 3$), we characterize the two important class of Weakly-Berwald $(\alpha, \beta)$--metrics of Scalar flag curvature. So first we have to prove the following lemma.

Lemma 3.1. Let $(M, F)$ be an $n$-dimensional Finsler manifold ($n \geq 3$). Suppose that $(\alpha, \beta)$--metrics $F = (\alpha + \beta)^2$ and $F = \sqrt{c_1 \alpha^2 + c_2 \alpha \beta + c_3 \beta^2}$ (where $c_1, c_2$ and $c_3$ are constants) is also similar. So we omit it. By a direct computation, we have

$$\Phi = -\frac{A \varphi}{(1 - K s^2) s'}.$$

where

$$\varphi = 1 + 2s + s^2,$$  

$$A = -12ns^3 + 6(1 + n)s^2 + 4n(1 + 2b^2) + 4(1 - b^2) - 2(n + 1)(1 + 2b^2).$$
Assume that $\Phi = 0$. Then $A=0$. Multiplying $A=0$ with $a^2$ yields 

$$
(4\beta n(2b^2 + 1) - 4\beta (b^2 - 1))a^2 - 12n\beta^3 - a(2(n + 1)(2b^2 + 1) + 6\beta^2(n + 1)) = 0.
$$

Hence we have,

$$
2(n + 1)(2b^2 + 1) + 6\beta^2(n + 1) = 0.
$$

Clearly, observe that $\beta^3$ is not divisible by $a^2$. Since we’ve $k = 0$ by (3.1), which is a contradiction with $k \neq 0$. So $\Phi \neq 0$. By using this lemma 3.1, now we can prove the following

**Theorem 3.3.** Let $(M, F)$ be an $n$-dimensional Finsler manifold $(n \geq 3)$. Assume that $(\alpha, \beta)$ – metrics $F = (\alpha + \beta)^2$ and $F = \sqrt{c_1a^2 + c_2a\beta + c_3\beta^2}$ (where $c_1, c_2$ and $c_3$ are constants) are of scalar flag curvature $K = K(x, y)$. Then $F$ is weak Berwald metric if and only if $F$ is Berwald metric and $K = 0$. In this case, $F$ must be locally Minkowskian.

**Proof:** By the above lemma and (2.9) we know that $(\alpha, \beta)$ – metrics $F = (\alpha + \beta)^2$ and $F = \sqrt{c_1a^2 + c_2a\beta + c_3\beta^2}$ (where $c_1, c_2$ and $c_3$ are constants) can’t represents the Riemannian metrics, where $k \neq 0$ a constant and $\beta \neq 0$.

The sufficiency is obvious. We just prove the necessity. First, we assume that the metric $F$ is weak Berwald. By lemma(3.1), we know that $S = (n + I)c(x)F$ with $c(x) = 0$ and $r\theta \theta = 0,s_\theta = 0$. Further, by schur lemma[13], $F$ must be of constant flag curvature. From (3.2), we can simplify (2.8), (2.14) and (2.15) as follows

$$
G^i - G^i = aQs^i_\theta, I_i = \frac{-\Phi s^i_\theta}{2\Delta}, J = 0.
$$

In addition, from (2.9) we obtain

$$
I_i b^i := -\frac{\Phi (\varphi - s\Phi)}{2\Delta} (b^2 - s^2).
$$

Thus (2.16) can be expressed as follows

$$
\frac{\Phi s^i_\theta}{2\Delta a} a^i k s^k_\theta + \frac{\Phi s^i_\theta}{2\Delta a} (sQs^i_\theta + Q's^i_\theta (b^2 - s^2))
$$

$$
- KF \frac{\Phi}{2\Delta} (\varphi - s\varphi) (b^2 - s^2) = 0.
$$

By lemma(3.1), we’ve

$$
s_\theta s^i_\theta \Delta - K\alpha^2 \varphi (\varphi - s\varphi) (b^2 - s^2) = 0. \tag{3.3}
$$

**Case I:** $F = \frac{(\alpha + \beta)^2}{\alpha}$. In this case,

$$
\Delta = \frac{\varphi (1 + 2b^2 - 3s^2)}{(s - I)^3}.
$$

Then (3.3) becomes

$$
(1 + 2b^2 - 3s^2) s_\theta s^i_\theta - K \alpha^2 (b^2 - s^2) (s - I)^3 = 0.
$$

Multiplying this equation with $\alpha^6$ yields

$$
Kb^6 \alpha^8 + \{(1 + 2b^2) s_\theta s^i_\theta - K \beta^2 (I + 3k^2 b^2) \alpha^6
$$

$$
+ 3\beta^2 (k\beta^2 (I + k^2) + s_\theta s^i_\theta) \alpha^4
$$

$$
- K\beta^6 (3 + b^2) \alpha^2 = -K\beta^6. \tag{3.4}
$$

Note that, the left of (3.4) is divisible by $\alpha^2$. Hence we can obtain that the flag curvature $K = 0$, because $k \neq 0$ and $\beta^6$ is not divisible by $\alpha^2$. Substituting $K = 0$ into (3.3), we’ve $s^i_\theta = 0$, i.e., $\beta$ is closed. By (3.2), we know that $\beta$ is parallel with respect to $\alpha$. Then $F = \frac{(\alpha + \beta)^2}{\alpha}$ is a Berwald metric, where $k \neq 0$ a constant. Hence $F$ must be locally Minkowskian.
Case II: $F = \sqrt{c_i a^2 + c_i a \beta + c_i \beta^2}$ (where $c_i, c_j$ and $c_k$ are constants). In this case,

$$
\Delta = \frac{4c_i^2 + 6c_i c_j s_i + 3c_i^2 s^2 + 2c_i c_j s^3 - c_i^2 b^2 + 4b^2 c_i c_j}
{(2c_i + c_j s)^2}
$$

Then (3.3) become

$$
(4c_i^2 + 3c_i^2 s^2 - b^2 c_i^2 + 2c_i c_j s^3 + 6c_i c_j s + 4b^2 c_i c_j) s_i s_0 = \alpha^2 (b^2 - s^2) (2c_i + c_j s)^3 = 0.
$$

$$
\implies A + \alpha B = 0, \text{ where }
$$

$$
A = [-12K c_i c_j b - 8K c_i b] b
- 8K c_i b c_i b a^4 + [6s_i s_i s_0] c_i c_j b
- (K c_i b^2 b - 6K c_i c_j b^2) b
- (6K c_i c_j b^2 b - 12K c_i c_j b^2 b
- 2s_i s_i s_0 s_i c_i b^2 + K c_i b^2 b]
$$

$$
B = -8K c_i b^2 a^4 + [s_i s_i s_i (-b^2 c_i^2 + 4c_i c_i) + c_i b^2]) - 6K b^2 c_i c_j s_i b
- 2c_i b - (12K b c_i c_j b c_i b
- 8K c_i c_j b) a^2 + [3s_i s_i b^2 c_i^2 b^2 + K b^4 c_i b - (K b^4 c_i b
- 6K b^4 c_i b] b]
$$

Obviously, we have $A = 0$ and $B = 0$. By $A = 0$ and clearly note that $\beta^3$ is not divisible by $a^2$. Then we obtain $s_i s_i s_i = 0$. Hence $\beta$ is closed. By (3.2), we know that $\beta$ is parallel with respect to $a$. Then $F$ is a Berwald metric. From (3.3), we find that $K = 0$. Hence $F$ is locally Minkowskian.

References


