Case Study of the Group $GL(n, \mathbb{Z})$

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Abstract: In this Article, we have discussed some of the properties of the infinite non-abelian group $GL(n, \mathbb{Z})$, $n \geq 2$. Such as the number of elements of order 2, number of subgroups of order 2 in this group and embedding of the $GL(n, \mathbb{Z})$ in $GL(m, \mathbb{Z}) \forall m \geq n$. Moreover for every finite group $G$, there exists $k \in \mathbb{N}$ such that $GL(k, \mathbb{Z})$ has a subgroup isomorphic to the group $G$.

Keywords: Infinite non-abelian group, $GL(n, \mathbb{Z})$

Notations: $GL(n, \mathbb{Z}) = \{ A_{ij} \in \mathbb{Z} : \det(A_{ij}) = \pm 1 \}$

Theorem 1: $GL(n, \mathbb{Z})$ can be embedded in $GL(m, \mathbb{Z}) \forall m \geq n$.

Proof: If we prove this theorem for $m = n + 1$, then we are done.

Let us define a mapping $\varphi : GL(n, \mathbb{Z}) \rightarrow GL(n + 1, \mathbb{Z})$

Such that $\varphi \left( \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \right) = \begin{bmatrix} a_{11} & \cdots & a_{1n} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}$

Let $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nn} \end{bmatrix}$

Such that determinant of A and B is $\pm 1$, where $a_{ij}, b_{ij} \in \mathbb{Z}$.

Then $A.B = \begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nn} \end{bmatrix}$

Where $c_{ij} = a_{ij}b_{ij} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$

$\Rightarrow \varphi(A.B) = \begin{bmatrix} c_{11} & \cdots & c_{1n} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ c_{n1} & \cdots & c_{nn} & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}$

$(1)$

$\Rightarrow \varphi(A).\varphi(B) = \begin{bmatrix} a_{11} & \cdots & a_{1n} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} b_{11} & \cdots & b_{1n} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ b_{n1} & \cdots & b_{nn} & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}$
\[
\begin{bmatrix}
  c_{11} & \ldots & c_{1n} \\
  \vdots & \ddots & \vdots \\
  c_{n1} & \ldots & c_{nn}
\end{bmatrix}
\]

\[= \begin{bmatrix}
  c_{11} & \ldots & c_{1n} \\
  \vdots & \ddots & \vdots \\
  0 & \ldots & 0
\end{bmatrix} \quad \text{(2)}
\]

Now from (1) and (2) one can easily say that \( \varphi \) is homomorphism.

Now consider \( \ker \varphi \),

\[
\ker \varphi = \left\{ A \in GL(n, \mathbb{Z}) : \varphi(A) = I_{(n+1) \times (n+1)} \right\} \quad \text{where } I \text{ is identity matrix}
\]

Clearly \( \ker \varphi = \{ I_{n \times n} \} \).

Hence \( \varphi \) is injective homomorphism.

So, by fundamental theorem of isomorphism one can easily conclude that \( GL(n, \mathbb{Z}) \) can be embedded in \( GL(m, \mathbb{Z}) \) for all \( m \geq n \).

**Theorem 2:** \( GL(n, \mathbb{Z}) \) is non-abelian infinite group \( n \geq 2 \).

**Proof:** Consider \( GL(2, \mathbb{Z}) = \left\{ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{Z} \text{ and } \det(A) = \pm 1 \right\} \)

Consider \( A_c = \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix} \) such that \( c \in \mathbb{Z} \)

And let \( H = \{ A_c : c \in \mathbb{Z} \} \)

Clearly \( H \) is subset of \( GL(2, \mathbb{Z}) \).

And \( H \) has infinite elements.

Hence \( GL(2, \mathbb{Z}) \) is infinite group.

Now consider two matrices \( \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \) and \( \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} \in GL(2, \mathbb{Z}) \)

\[
\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix}
\]

And

\[
\begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 7 \\ 3 \end{bmatrix}
\]

Hence, \( GL(2, \mathbb{Z}) \) is non-abelian.

And by a direct application theorem 1, one can conclude that \( GL(n, \mathbb{Z}) \) is infinite and non-abelian \( \forall n \geq 2 \).

**Theorem 3:** Number of elements of order 2 in \( GL(n, \mathbb{Z}) \) is infinite and hence number of subgroups of order 2 is infinite.

**Proof:** Consider \( GL(2, \mathbb{Z}) = \left\{ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{Z} \text{ and } \det(A) = \pm 1 \right\} \)

Let \( A \in GL(2, \mathbb{Z}) \), and order of \( A \) is 2.
Then \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) for some \( a, b, c, d \in \mathbb{Z} \) and \( A^2 = I \)

\[
\Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

\[
\Rightarrow \begin{bmatrix} a^2 + bc & ab + bd \\ ca + cd & bc + d^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

\[
\Rightarrow a^2 + bc = 1, ab + bd = 0, ca + cd = 0, bc + d^2 = 1
\]

Since \( a^2 + bc = 1 \), then:

Case (i): \( a^2 = 1, bc = 0 \)

\[
\Rightarrow a = \pm 1 \text{ and either } b \text{ or } c = 0
\]

Subcase (i): If we take \( a = 1 \) and \( b = 0 \)

Then \( \begin{bmatrix} 1 & 0 \\ c + cd & d^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \)

\[
\Rightarrow c + cd = 0
\]

\[
\Rightarrow c(1 + d) = 0
\]

\[
\Rightarrow \text{either } c = 0 \text{ or } d = -1
\]

\[
\Rightarrow \text{if } c \neq 0 \text{ then } d = -1
\]

Let \( c \neq 0 \)

Then our case is \( a = 1, b = 0, c \neq 0, d = -1 \)

\[
\Rightarrow \begin{bmatrix} 1 & 0 \\ c & -1 \end{bmatrix} \text{ has order 2 if } c \neq 0 \text{ and } c \in \mathbb{Z}
\]

Clearly \( c \) has infinite choices.

\[
\Rightarrow \text{Number of elements of order 2 in } GL(2, \mathbb{Z}) \text{ is infinite.}
\]

And since number of elements of order 2 = Number of subgroups of order 2 in any group.

\[
\Rightarrow \text{Number of subgroups of order 2 in } GL(2, \mathbb{Z}) \text{ is infinite.}
\]

Also by a direct application of theorem 1, one can easily conclude that this theorem is valid for every value of \( n \).

**Theorem 4:** Every finite group can be embedded in \( S_n \) for some \( n \in \mathbb{N} \)

**Proof:** Let \( G \) be any group and \( A(G) \) be the group of all permutations of set \( G \).

For any \( a \in G \), define a map \( f_a : G \rightarrow G \) such that \( f_a(x) = ax \)

Then as \( x = y \Rightarrow ax = ay \)

\[
\Rightarrow f_a(x) = f_a(y)
\]
Hence, $f_a$ is well defined.

Clearly $f_a$ is one-one.

Also for any $y \in G$, since $f_a(a^{-1} y) = y$.

$\Rightarrow f_a$ is onto.

And hence $f_a$ is permutation.

Let K be set of all such permutations.

Clearly K is subgroup of $A(G)$.

Now define a mapping $\varphi : G \rightarrow K$ such that $\varphi(a) = f_a$

Clearly $\varphi$ is well-defined and one-one map.

And consider the following equation

$$\varphi(a \cdot b) = f_{ab} = f_a \circ f_b = \varphi(a) \cdot \varphi(b)$$

Which shows that $\varphi$ is homomorphism.

Obviously, $\varphi$ is onto homomorphism

$\Rightarrow \varphi$ is isomorphism.

And hence the theorem.

**Theorem 5:** $S_n$ is isomorphic to some subgroup of $GL_n \mathbb{Z}$ for all $n \in \mathbb{N}$.

**Proof:** Let $S_n$ be the permutation group on n symbols.

Define $\varphi : SGL_n \rightarrow (\mathbb{Z})$ such that:

$$[\sigma]_{im} \in S_n$$

Where $[\sigma]_{im}$ is a permutation matrix obtained by $\sigma$ i.e. if $\sigma = \begin{pmatrix} 1 & 2 & \ldots & n \\ \beta_1 & \beta_2 & \ldots & \beta_n \end{pmatrix}^T$ then

$$[\sigma] = \begin{pmatrix} R_1 \\ R_2 \\ \vdots \\ R_n \end{pmatrix}$$

Where $R_i$ is a row of identity matrix.

Clearly $\varphi$ is a homomorphism.

Now consider the kernel of this homomorphism.

$\ker \varphi \ni I_{nm} \Rightarrow \beta_i \quad \forall i$

$\Rightarrow \ker \varphi$ is trivial.
Hence the homomorphism is injective. 
\[ S \] is isomorphic to some subgroup of \( GL_n(\mathbb{Z}) \) for all \( n \in \mathbb{N} \).

**Theorem 6:** For every finite group \( G \), there exists \( k \in \mathbb{N} \) such that \( GL(k, \mathbb{Z}) \) has a subgroup isomorphic to the group \( G \).

**Proof:** It is an obvious observation of theorem 4 and theorem 5.

**Conclusion**

\( GL(n, \mathbb{Z}) \) is non-abelian infinite group having infinite number of elements of order 2 as well as subgroups of order 2. Also \( GL(n, \mathbb{Z}) \) can be embedded in \( GL(m, \mathbb{Z}) \) for \( m \geq n \), Moreover for every finite group \( G \), there exists \( k \in \mathbb{N} \) such that \( GL(k, \mathbb{Z}) \) has a subgroup isomorphic to the group \( G \).

**References**

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