Interpolation By Inverse Cubic Spline Method

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Abstract — A formula for inverse cubic spline is derived by applying a method called reversion of series and its error is also analyzed.

Keywords — Natural cubic spline, Rolle’s Theorem, Series Reversion.

I. INTRODUCTION

The process of constructing a function say \( g(x) \) which satisfies the given set of data points \((x, f(x))\) is called interpolation. If the interpolated function is a polynomial it is known as a polynomial interpolation. A cubic spline consists of third degree polynomial bits joined together. [2], [3]

Consider a function \( y = f(x) \) satisfied by a set of data points \((a_i, y_i)\), \(i = 1, 2, ..., n\). Let \( I_i = [a_{i-1}, a_i], i = 2, 3, ..., n \) be a subinterval of \( I = [a, b] \). A cubic spline \( S_i(x) \) is a polynomial of degree 3 and having continuous derivatives up to order 2.

\[
S_i(x) = \begin{cases} S_1(x), & x < a_2 \\ S_2(x), & a_1 < x < a_3 \\ \vdots \\ S_n(x), & a_{n-1} < x < a_n \end{cases}
\]

We have, \( S_i(x) \) is the cubic spline polynomial of degree 3 and having continuous derivatives up to order 2.

The conditions for the natural cubic spline are

(i) \( S_i(x), S_2(x), ..., S_n(x) \) are all almost cubic

(ii) \( S_i(x) = y_i, i = 1, 2, ..., n \)

(iii) \( S_i(x), S_i'(x) and S_i''(x) \) are continuous in \( I = [a, b] \)

(iv) \( S_i''(a) = S_i''(b) = 0 \)

Since \( S(x) \) is a cubic spline, [1], [5] \( S''(x) \) is linear and continuous.

Let \( S_i^{-1}(x) = M_{i-1} \frac{a_i - x}{h_i} + M_i \frac{x - a_{i-1}}{h_i} \)

Integrating twice with respect to \( x \)

\[
S_i^{-1}(x) = M_{i-1} \left( \frac{a_i - x}{h_i} \right)^3 + M_i \left( \frac{x - a_{i-1}}{h_i} \right)^3 + C_i (a_i - x) + d_i (x - a_{i-1})
\]

Applying the conditions \( S_i^{-1}(x_{i-1}) = y_{i-1} \) and \( S_i(x_i) = y_i \)

\[
S_i^{-1}(x) = M_{i-1} \left( \frac{a_i - x}{h_i} \right)^3 + M_i \left( \frac{x - a_{i-1}}{h_i} \right)^3 + \left( y_{i-1} - \frac{M_{i-1} h_i^2}{6} \right) \left( \frac{a_i - x}{h_i} \right) + \left( y_i - \frac{M_i h_i^2}{6} \right) \left( \frac{x - a_{i-1}}{h_i} \right), x \in [a_{i-1}, a_i]
\]

\[
(1)
\]

Where \( h_i = a_i - a_{i-1}, i = 2, 3, ..., n \) which is the cubic spline equation.

The \( M_i \) are obtained by the equations:

\[
\begin{align*}
\mu_i M_{i-1} + 2M_i + \lambda_i M_{i+1} &= d_i, i = 1, 2, ..., n \\
2M_i + \lambda_i M_{x+1} &= d_i \\
\mu_n M_{n-1} + 2M_n &= d_n
\end{align*}
\]

Where \( \sigma_i = \frac{y_i - y_{i-1}}{h_i} \), \( i = 2, 3, ..., n \) \( (3) \)

\[
\begin{align*}
\lambda_i &= \frac{h_{i+1}}{h_i + h_{i+1}} \\
\mu_i &= 1 - \lambda_i \\
d_i &= \frac{6(\sigma_{i+1} - \sigma_i)}{h_i + h_{i+1}}, i = 2, 3, ..., n-1
\end{align*}
\]

\[
(4)
\]
II. SERIES REVERSION METHOD

Method of series reversion [6] can be applied to the cubic polynomial
\[ y = a_0 + a_1x + a_2x^2 + a_3x^3 \]  
(5)

Then
\[ Y = a_1x + a_2x^2 + a_3x^3 \]  
(6)

where
\[ Y = y - a_0 \]

Inverting the above
\[ x = A_1Y + A_2Y^2 + A_3Y^3 \]  
(7)

Substituting (6) in (7), and equating we get,
\[ A_1 = \frac{1}{a_1} \]
\[ A_2 = \frac{-a_2}{a_3} \]
\[ A_3 = \frac{2a_2^2 - a_3a_1}{a_1^5} \]

Hence the inverted cubic in Y is
\[ x = \frac{1}{a_1} Y + \frac{-a_2}{a_1} Y^2 + \frac{2a_2^2 - a_3a_1}{a_1^5} Y^3 \]
\[ x = \frac{1}{a_1} (y - a_0) + \frac{-a_2}{a_1} (y - a_0)^2 + \frac{2a_2^2 - a_3a_1}{a_1^5} (y - a_0)^3 \]

\[ x = \frac{1}{a_1} y + -\frac{a_2}{a_1} Y^2 + \frac{2a_2^2 - a_3a_1}{a_1^5} Y^3 \]  
(8)

Where
\[ a_i - x = X_i \]
\[ x - a_{i-1} = h_i - X_i \]

\[ S_{i-1}(x) = M_{i-1} \frac{X_i^3}{6h_i} + M_i \left( \frac{h_i - X_i}{6h_i} \right)^3 + \]
\[ \left( y_{i-1} - \frac{M_{i-1}h_i^2}{6} \right) \left( \frac{a_i - x}{h_i} \right) + \]
\[ \left( y_i - \frac{M_i h_i^2}{6} \right) \left( \frac{x - a_{i-1}}{h_i} \right) \]  
(9)

Hence the Natural cubic spline is
\[ S_{i-1}(x) = y_i + p_i X_i + q_i X_i^2 + r_i X_i^3, \]
\[ i = 2, 3, \ldots, n \]  
(10)

By applying reversion of series method to the above equation inverse cubic spline is
\[ S_{i-1}^{-1}(y) = a_i - \left[ \frac{1}{p_i} (y - y_i) - \frac{q_i}{p_i^3} (y - y_i)^2 \right] \]
\[ - \frac{2q_i^2 - p_i r_i}{p_i^5} (y - y_i)^3 \]  
(11)

Where \( y \in [y_{i-1}, y_i], i = 2, 3, \ldots, n \)  
(11)
IV. ERROR ANALYSIS OF INVERSE CUBIC SPLINE

Theorem:

Let \( S_n^{-1}(y) \in \pi_n \) (\( \pi_n \) - set of all polynomials not exceeding \( n \)), which interpolates at \( f^{-1}(y) \) at \( (n+1) \) distinct points say \( y_0, y_1, \ldots, y_n \in [c,d] \).
Then \( \forall y \in [c,d] \) there exist a point \( e_n \in (c,d) \) such that
\[
 f^{-1}(y) - S_n^{-1}(y) = \frac{1}{(n+1)} \left( \frac{d^{n+1}}{dy^{n+1}} f^{-1}(e_n) \right) \prod_{j=0}^{n} (y - y_j) 
\]
(11)

Proof:

Given \( S_n^{-1}(y) \in \pi_n \) is the approximating polynomial of \( f^{-1}(y) \) at \((n+1)\) points say \( y_0, y_1, \ldots, y_n \). If \( y \) is one of the interpolation point then LHS and RHS of (11) are both zero. Therefore (11) is satisfied trivially.

Suppose \( y \neq y_j, 0 \leq j \leq n \) then \( \prod_{j=0}^{n} (y - y_j) \neq 0 \)

So let \( q(y) = \prod_{j=0}^{n} (y - y_j) \)

Construct a function
\[
 F(y) = f^{-1}(y) - S_n^{-1}(y) - \lambda q(y) \text{ where } \lambda \text{ is chosen so that } F(y) = 0.
\]

Therefore
\[
 \lambda = \frac{f^{-1}(y) - S_n^{-1}(y)}{q(y)}
\]

Since \( y \neq y_j \) gives \( q(y) \neq 0 \), \( \lambda \) is well defined.

Since \( F(y) \) is a function which is \( (n+1) \) times differentiable in \([c,d]\) and \( F(y) \) vanishes at \((n+1)\) points \( y_0, y_1, \ldots, y_n \) in \((c,d)\) by Rolle’s theorem[4] there exist a number \( e_n \) in \((c,d)\) such that
\[
 \frac{d^{n+1}}{dy^{n+1}} F(e_n) = 0
\]

Hence
\[
 0 = \frac{d^{n+1}}{dy^{n+1}} F(e_n) = \frac{d^{n+1}}{dy^{n+1}} \left( f^{-1}(e_n) \right) - \lambda \frac{d^{n+1}}{dy^{n+1}} \left( q(e_n) \right)
\]

Hence the inverse cubic spline is
\[ S_1^{-1}(y) = -11.28y^3 + 19.81y^2 - 10.472y \\
 + 1.943, \quad y \in [0.5, 0.5477] \]

\[ S_2^{-1}(y) = 3.75y^3 - 6.225y^2 + 4.635y \\
 - 0.993, \quad y \in [0.5477, 0.6245] \]

\[ S_3^{-1}(y) = -4.18y^3 + 9.6612y^2 - 5.972y \\
 + 1.3694, \quad y \in [0.6245, 0.6708] \]

\[ S_4^{-1}(y) = 8.847y^3 - 19.32y^2 + 15.493y \\
 - 3.923, \quad y \in [0.6708, 0.7280] \]

REFERENCE


