A general Eulerian integral associated with product of

two multivariable Aleph-functions

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ABSTRACT

The present paper is evaluated a new Eulerian integral associated with the product of two multivariable Aleph-functions, the multivariable A-function and a generalized hypergeometric function with general arguments. We will study the cases concerning the multivariable I-function defined by Sharma et al [3] and Srivastava-Daoust polynomial [4].

Keywords: Eulerian integral, multivariable I-function, generalized hypergeometric function of several variables, multivariable Aleph-function, multivariable A-function, generalized hypergeometric function.

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1. Introduction

In this paper, we consider a general class of Eulerian integral concerning the product of two Multivariable Aleph-functions, the multivariable A-function and a generalized hypergeometric function.

The Aleph-function of several variables is an extension of the multivariable I-function defined by Sharma and Ahmad [3], itself is an generalisation of G and H-functions of several variables defined by Srivastava et al [7]. The serie representation of the multivariable A-function is given by Gautam [1] as

\[
A[u_1, \ldots, u_v] = A^{0, \lambda, (\alpha^{(u)}, \beta^{(u)}, \cdots, \alpha^{(v)}, \beta^{(v)})}_{\lambda, \gamma, (M^{(u)}, N^{(u)}, \cdots, M^{(v)}, N^{(v)})} \left( \begin{array}{c} u_1 \\ \vdots \\ u_v \\ \end{array} \right) \left( \begin{array}{c} \{ (g_j) ; (\gamma^{(u)}_j, \cdots, \gamma^{(v)}_j) \} ; A \\ \vdots \\ \{ (f_j) ; (\xi^{(u)}_j, \cdots, \xi^{(v)}_j) \} ; C \end{array} \right)
\]

\[
(q^{(1)}, \theta^{(1)})_{1, M^{(1)}} ; \cdots ; (p^{(v)}, \theta^{(v)})_{1, M^{(v)}} 
\]

\[
(p^{(1)}, e^{(1)})_{1, N^{(1)}} ; \cdots ; (p^{(v)}, e^{(v)})_{1, N^{(v)}} 
\]

= \sum_{G_1=1}^{a_1} \sum_{g_1=1}^{\infty} \phi_1 \prod_{i=1}^{v} \phi_i \gamma_i^{\alpha_i}_i \eta_i^{\alpha_i}_i \frac{(-)^{\sum_{i=1}^{v} g_i}}{\prod_{i=1}^{v} \xi_i^{\alpha_i}_i g_i!} (1.1)

where

\[
\phi_1 = \frac{\prod_{j=1}^{\lambda} \Gamma \left( 1 - g_j + \sum_{i=1}^{\nu} \gamma^{(i)} \eta_{G_i, g_i} \right)}{\prod_{j=\lambda+1}^{\Lambda} \Gamma \left( g_j - \sum_{i=1}^{\nu} \gamma^{(i)} U_i \right) \prod_{j=1}^{C} \Gamma \left( 1 - f_j + \sum_{i=1}^{\nu} \epsilon^{(i)} \eta_{G_i, g_i} \right)} (1.2)
\]

\[
\phi_i = \frac{\prod_{j=1, j \neq m_i}^{\alpha_i} \Gamma \left( p_j^{(i)} - \epsilon_j^{(i)} \eta_{G_i, g_i} \right) \prod_{j=\alpha_i+1}^{\beta_i} \Gamma \left( 1 - q_j^{(i)} + \eta_j^{(i)} \eta_{G_i, g_i} \right)}{\prod_{j=\alpha_i+1}^{\alpha_i+1} \Gamma \left( 1 - p_j^{(i)} + \epsilon_j^{(i)} \eta_{G_i, g_i} \right) \prod_{j=\beta_i+1}^{\gamma_i} \Gamma \left( 1 - q_j^{(i)} + \eta_j^{(i)} \eta_{G_i, g_i} \right)} ; i = 1, \cdots, v (1.3)
\]

and

\[
\eta_{G_i, g_i} = \frac{p_j^{(i)} + g_i}{\epsilon_j^{(i)}} , i = 1, \cdots, v (1.4)
\]

which is valid under the following conditions:

\[
\epsilon_j^{(i)} \left( p_j^{(i)} + p_i^{(i)} \right) \neq \epsilon_j^{(i)} \left( p_m^{(i)} + g_i \right) (1.5)
\]

and
\[ u_i \neq 0, \sum_{j=1}^{A} \gamma_j^{(i)} - \sum_{j=1}^{C} \epsilon_j^{(i)} + \sum_{j=1}^{M^{(i)}} \eta_j^{(i)} - \sum_{j=1}^{N^{(i)}} \zeta_j^{(i)} < 0, i = 1, \cdots, v \]  

(1.6)

Here, \( \lambda, A, C, \alpha_i, \beta_i, m_i, n_i \in \mathbb{N}^*; i = 1, \cdots, v; f_j, g_j, \delta_j^{(i)}, \epsilon_j^{(i)}, \gamma_j^{(i)}, \eta_j^{(i)}, \zeta_j^{(i)} \in \mathbb{C} \)

We have:

\[
\set{\eta}{z_1, \cdots, z_r} = \mathbb{N}_{p, q, \tau, r; \pi, q, \tau, r; \tau, r; \pi, q, \tau, r; \tau, r; \pi, q, \tau, r; \tau, r; \pi, q, \tau, r; \tau, r} \left( \begin{array}{c}
\tau_1 \\
\vdots \\
\tau_r
\end{array} \right)
\]

\[
\left[ (a_1^{(r)}, \alpha_1^{(r)}, \cdots, \alpha_r^{(r)})_{1, n_1}, \tau_1(a_1^{(r)}, \alpha_1^{(r)}, \cdots, \alpha_r^{(r)})_{n_1+1, p_1};\right.
\]

\[
\left. \cdots \cdots \right]
\]

\[
\left[ (\theta_1^{(r)}, \delta_1^{(r)}, \eta_1^{(r)})_{1, m_1}, \tau_1(\theta_1^{(r)}, \delta_1^{(r)}, \eta_1^{(r)})_{m_1+1, q_1}; \right.
\]

\[
\left. \cdots \cdots \right]
\]

\[
= \frac{1}{(2\pi \omega)^r} \int_{L_1} \cdots \int_{L_r} \psi(s_1, \cdots, s_r) \prod_{k=1}^{r} \theta_k(s_k) z_k^{s_k} ds_1 \cdots ds_r
\]

(1.8)

with \( \omega = \sqrt{-1} \)

\[
\psi(s_1, \cdots, s_r) = \frac{\prod_{j=1}^{n} \Gamma(1 - a_j + \sum_{k=1}^{r} \alpha_j^{(k)} s_k)}{\sum_{i=1}^{R} \prod_{j=n+1}^{p_i} \Gamma(a_{ji} - \sum_{k=1}^{r} \alpha_{ji}^{(k)} s_k) \prod_{j=1}^{q_i} \Gamma(1 - b_{ji} + \sum_{k=1}^{r} \beta_{ji}^{(k)} s_k)}
\]

(1.9)

and

\[
\theta_k(s_k) = \frac{\prod_{j=1}^{m_k} \Gamma(c_j^{(k)} - d_j^{(k)} s_k) \prod_{j=1}^{n_k} \Gamma(1 - c_j^{(k)} + \gamma_j^{(k)} s_k)}{\sum_{i=1}^{R^{(k)}} \prod_{j=m_k+1}^{p_i} \Gamma(1 - d_{ji}^{(k)} + \delta_{ji}^{(k)} s_k) \prod_{j=n_k+1}^{q_i} \Gamma(c_{ji}^{(k)} - \gamma_{ji}^{(k)} s_k)}
\]

(1.10)

Suppose, as usual, that the parameters

\[ a_j, j = 1, \cdots, p; b_j, j = 1, \cdots, q; \]

\[ c_j^{(k)}, j = 1, \cdots, n_k; c_{ji}^{(k)}, j = n_k + 1, \cdots, p_i^{(k)}; \]

\[ d_j^{(k)}, j = 1, \cdots, m_k; d_{ji}^{(k)}, j = m_k + 1, \cdots, q_i^{(k)}; \]

with \( k = 1, \cdots, r, i = 1, \cdots, R, z^{(k)} = 1, \cdots, R^{(k)} \)

are complex numbers, and the \( \alpha', \beta', \gamma' \) and \( \delta' \)'s are assumed to be positive real numbers for standardization purpose such that

\[
U_i^{(k)} = \sum_{j=1}^{n} \alpha_j^{(i)} + \tau_i \sum_{j=n+1}^{p_i} \alpha_{ji}^{(i)} + \sum_{j=1}^{m_k} \gamma_j^{(i)} + \tau_i \sum_{j=n_k+1}^{p_i^{(k)}} \gamma_{ji}^{(i)} \sum_{j=1}^{q_i^{(k)}} \beta_{ji}^{(i)} - \sum_{j=1}^{m_k} \delta_j^{(i)}
\]

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The reals numbers \( \tau_i \) are positives for \( i = 1 \) to \( R \), \( \tau_{i(k)} \) are positives for \( i^{(k)} = 1 \) to \( R^{(k)} \).

The contour \( L_k \) is in the \( s_k \)-plane and run from \( \sigma - i \infty \) to \( \sigma + i \infty \) where \( \sigma \) is a real number with loop, if necessary, ensure that the poles of \( \Gamma(d_j^{(k)} - \delta_j^{(k)} s_k) \) with \( j = 1 \) to \( m_k \) are separated from those of \( \Gamma(1 - a_j + \sum_{i=1}^{n} \alpha_j^{(k)} s_k) \) with \( j = 1 \) to \( n \) and \( \Gamma(1 - c_j^{(k)} + \gamma_j^{(k)} s_k) \) with \( j = 1 \) to \( n_k \) to the left of the contour \( L_k \). The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as:

\[
|\arg z_k| < \frac{1}{2} A^{(k)}_1 \pi, \quad \text{where}
\]

\[
A^{(k)}_1 = \sum_{j=1}^{n} \alpha_j^{(k)} - \tau_i - \sum_{j=n+1}^{p_i} \alpha_j^{(k)} - \tau_i - \sum_{j=1}^{q_i} \beta_j^{(k)} + \sum_{j=1}^{\tau_i} \gamma_j^{(k)} - \tau_i \sum_{j=n_k+1}^{p_i} \gamma_j^{(k)} \quad (1.12)
\]

The complex numbers \( z_k \) are not zero. Throughout this document, we assume the existence and absolute convergence conditions of the multivariable Aleph-function.

We may establish the asymptotic expansion in the following convenient form:

\[
N(z_1, \ldots, z_r) = 0 \left( |z_1|^{\alpha_1}, \ldots, |z_r|^{\alpha_r} \right) \max(|z_1|, \ldots, |z_r|) \to 0
\]

\[
N(z_1, \ldots, z_r) = 0 \left( |z_1|^{\beta_1}, \ldots, |z_r|^{\beta_r} \right) \min(|z_1|, \ldots, |z_r|) \to \infty
\]

where \( k = 1, \ldots, r : \alpha_k = \min[\Re(d_j^{(k)} / \delta_j^{(k)})], j = 1, \ldots, m_k \) and

\[
\beta_k = \max[\Re((c_j^{(k)} - 1) / \gamma_j^{(k)})], j = 1, \ldots, n_k
\]

We will use the following notations in this paper

\[
U = p_i, q_i, \tau_i; R \quad V = m_1, n_1; \ldots; m_r, n_r \quad (1.13)
\]

\[
W = p_{i(1)}, q_{i(1)}, \tau_{i(1)}; R^{(1)}, \ldots, p_{i(r)}, q_{i(r)}, \tau_{i(r)}; R^{(r)} \quad (1.14)
\]

\[
A = \{(a_j ; \alpha_j^{(1)}, \ldots, \alpha_j^{(r)}), \{\tau_k ; (a_j, \alpha_j^{(1)}, \ldots, \alpha_j^{(r)})_{n+1, p_i}\} \quad (1.15)
\]

\[
B = \{\tau_k ; (b_j, \beta_j^{(1)}, \ldots, \beta_j^{(r)})_{m+1, q_i}\} \quad (1.16)
\]

\[
C = \{(c_j^{(1)} ; \gamma_j^{(1)})_{1, n_1}, \tau_{i(1)} ; (c_j^{(1)} ; \gamma_j^{(1)}, n_{1+1, p_i} ; \gamma_{i(1)} ; (c_j^{(r)} ; \gamma_j^{(1)})_{n_{1+1, p_i}} \} \quad (1.17)
\]

\[
D = \{(d_j^{(1)} ; \delta_j^{(1)}), \tau_{i(1)} ; (d_j^{(1)} ; \delta_j^{(1)}_{m_1+1, q_i} ; \delta_{i(1)} ; (d_j^{(r)} ; \delta_j^{(1)}_{m_1+1, q_i})_{m_r+1, q_{i(r)}} \} \quad (1.18)
\]

The multivariable Aleph-function write:
Consider the Aleph-function of s variables

\[
\mathcal{N}(z_1, \ldots, z_s) = \mathcal{N}^{0,n',m'_1,n'_2, \ldots, m'_s,n'_r}_{\mu_1', \mu_2', \ldots, \mu_s'} A : C \begin{pmatrix} z_1 \\ \vdots \\ \vdots \\ z_s \\ B : D \end{pmatrix} 
\]

(1.19)

with

\[
\begin{pmatrix} (u_1; \mu_j^{(1)}), \ldots, (u_1; \mu_j^{(r')})_{1,n'} \\ \vdots \\ (u_s; \mu_j^{(1)}), \ldots, (u_s; \mu_j^{(r')})_{n'+1,p'} \end{pmatrix}; \\
\begin{pmatrix} (v_1; \mu_j^{(1)}), \ldots, (v_1; \mu_j^{(r')})_{m'+1,q'} \\ \vdots \\ (v_s; \mu_j^{(1)}), \ldots, (v_s; \mu_j^{(r')})_{m'+1,q'} \end{pmatrix} 
\]

\[
= \frac{1}{(2\pi\omega)^s} \int_{L_1'} \cdots \int_{L_s'} \zeta(t_1, \ldots, t_s) \prod_{k=1}^s \phi_k(t_k) z_k^{t_k} \, dt_1 \cdots dt_s 
\]

(1.20)

with \( \omega = \sqrt{-1} \)

\[
\zeta(t_1, \ldots, t_s) = \frac{\prod_{j=1}^{n'} \Gamma(1 - u_j + \sum_{k=1}^s \mu_j^{(k)} t_k)}{\prod_{j=1}^{n'} \Gamma(1 - v_j + \sum_{k=1}^s \mu_j^{(k)} t_k)} 
\]

(1.21)

and \( \phi_k(t_k) = \frac{\prod_{j=1}^{n'_k} \Gamma(b_j^{(k)} t_k)}{\prod_{j=1}^{n'_k} \Gamma(1 - b_j^{(k)} t_k)} \frac{\prod_{j=1}^{n'_k} \Gamma(1 - a_j^{(k)} t_k)}{\prod_{j=1}^{n'_k} \Gamma(1 - b_j^{(k)} t_k) + \beta_j^{(k)} t_k}} \frac{\prod_{j=1}^{n'_k} \Gamma(1 - a_j^{(k)} t_k)}{\prod_{j=1}^{n'_k} \Gamma(1 - a_j^{(k)} s_k)}} 
\]

(1.22)

Suppose, as usual, that the parameters

\[
u_j, j = 1, \ldots, p'; u_j, j = 1, \ldots, q'; \\
\alpha_j^{(k)}, j = 1, \ldots, n'_k; \alpha_j^{(k)}, j = n_k + 1, \ldots, p'_k; \\
b_j^{(k)}, j = m'_k + 1, \ldots, q'_k; b_j^{(k)}, j = 1, \ldots, n'_k; \\
with k = 1, \ldots, s, i = 1, \ldots, r', r^{(k)} = 1, \ldots, r^{(k)}
\]

are complex numbers, and the \( \alpha', \beta', \gamma', s \) and \( \delta' \) are assumed to be positive real numbers for standardization purpose such that
The real numbers \( \tau_i \) are positives for \( i = 1, \ldots, s \), \( \nu_i^{(k)} \) are positives for \( i^{(k)} = 1 \ldots r^{(k)} \).

The contour \( L_k \) is in the \( t_k \)-p lane and run from \( \sigma - i\infty \) to \( \sigma + i\infty \) where \( \sigma \) is a real number with loop, if necessary, ensure that the poles of \( \Gamma(l_j^{(k)} - \beta_j^{(k)} t_k) \) with \( j = 1 \) to \( m_k^{(k)} \) are separated from those of \( \Gamma(1 - u_j + \sum_{i=1}^{s} l_j^{(k)} t_k) \) with \( j = 1 \) to \( N \) and \( \Gamma(1 - \alpha_j^{(k)} + \alpha_j^{(k)} t_k) \) with \( j = 1 \) to \( n_k^{(k)} \) to the left of the contour \( L_k \). The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as:

\[
|\arg z_k| < \frac{1}{2} B_i^{(k)} \pi, \quad \text{where}
\]

\[
B_i^{(k)} = \sum_{j=1}^{n_i^{(k)}} \mu_j^{(k)} - \nu_i^{(k)} \sum_{j=n_i^{(k)}+1}^{n_i^{(k)}} \mu_j^{(k)} - \nu_i^{(k)} \sum_{j=n_i^{(k)}+1}^{n_i^{(k)}} \nu_j^{(k)} + \sum_{j=1}^{n_i^{(k)}} \alpha_j^{(k)} - \nu_i^{(k)} \sum_{j=1}^{n_i^{(k)}} \alpha_j^{(k)} + \sum_{j=n_i^{(k)}+1}^{n_i^{(k)}} \beta_j^{(k)} \sum_{j=m_i^{(k)}+1}^{n_i^{(k)}} \beta_j^{(k)} > 0, \quad \text{with } k = 1, \ldots, s, i = 1, \ldots, r, i^{(k)} = 1, \ldots, r^{(k)} \tag{1.24}
\]

The complex numbers \( z_i \) are not zero. Throughout this document, we assume the existence and absolute convergence conditions of the multivariable Aleph-function.

We may establish the asymptotic expansion in the following convenient form:

\[
\mathbb{N}(z_1, \ldots, z_s) = 0(\max(|z_1|, \ldots, |z_s|)) \rightarrow 0 \tag{1.25}
\]

\[
\mathbb{N}(z_1, \ldots, z_s) = 0(\min(|z_1|, \ldots, |z_s|)) \rightarrow \infty
\]

where \( k = 1, \ldots, z : \alpha_j^{(k)} = \min\{|Re(b_j^{(k)}/\beta_j^{(k)}|)\}, j = 1, \ldots, m_k^{(k)} \text{ and } \beta_j^{(k)} = \max\{|Re((a_j^{(k)} - 1)/\alpha_j^{(k)}|)\}, j = 1, \ldots, n_k^{(k)} \)

We will use these following notations in this paper

\[
U' = p_i', q_i', \nu_i'; r'; r' = m_i^{(k)}, n_i^{(k)}; m_s^{(k)}, n_s^{(k)} \tag{1.26}
\]

\[
W' = p_i^{(1)}, q_i^{(1)}, \nu_i^{(1)}; r^{(1)}, \ldots, p_i^{(n)}, q_i^{(n)}, \nu_i^{(n)}; r^{(n)} \tag{1.27}
\]

\[
A' = \{(u_j; \mu_j^{(1)}, \ldots, \mu_j^{(s)}), \nu_i^{(1)}(u_{j1}; \mu_j^{(1)}), \ldots, \mu_j^{(s)})(n_i^{(1)}+1, p_i^{(1)})\} \tag{1.28}
\]

\[
B' = \{(u_{j1}; u_{j2}, \ldots, u_{jn}), v_{j1}, \ldots, v_{jn}(m_i^{(n)}+1, q_i^{(n)})\} \tag{1.29}
\]

\[
C' = (a_j^{(1)}; \alpha_j^{(1)}), \nu_i^{(1)}(a_j^{(1)}; \alpha_j^{(1)}), n_i^{(1)}+1, p_i^{(1)}, \ldots, (a_j^{(s)}; \alpha_j^{(s)}), \nu_i^{(n)}(a_j^{(s)}; \alpha_j^{(s)}), n_i^{(n)}+1, p_i^{(n)} \tag{1.29}
\]
The multivariable Aleph-function write:

\[ D' = (b_1^{(1)}; \beta_1^{(1)}, \ldots, b_{l+1}^{(1)}; \beta_{l+1}^{(1)})_{m_1, 1, \ldots, l_{(1)}}, (b_1^{(2)}; \beta_1^{(2)}, \ldots, b_{l+1}^{(2)}; \beta_{l+1}^{(2)})_{m_2, 1, \ldots, l_{(2)}}, \ldots, (b_1^{(s)}; \beta_1^{(s)}, \ldots, b_{l+1}^{(s)}; \beta_{l+1}^{(s)})_{m_s, 1, \ldots, l_{(s)}} \]

\[ (1.30) \]

The multivariable Aleph-function write:

\[ \mathcal{H}(z_1, \ldots, z_s) = \mathcal{H}_{U; W'}^{V_0; n'} \left( \begin{array}{c} z_1 \\ \vdots \\ z_s \end{array} \right) \begin{bmatrix} A' & : C' \\ \ddots & \ddots \end{bmatrix} \]

\[ (1.31) \]

2. Integral representation of generalized Lauricella function of several variables

The following generalized hypergeometric function in terms of multiple contour integrals is also required [6, page 39 eq.30]

\[ \frac{\Pi_{j=1}^{P} \Gamma(A_j)}{\Pi_{j=1}^{Q} \Gamma(B_j)} \, _pF_Q \left( (A_P); (B_Q); -(x_1 + \cdots + x_r) \right) \]

\[ = \frac{1}{(2\pi i)^r} \int_{L_1} \cdots \int_{L_r} \prod_{j=1}^{P} \Gamma(A_j + s_1 + \cdots + s_r) \prod_{j=1}^{Q} \Gamma(B_j + s_1 + \cdots + s_r) \Gamma(-s_1) \cdots \Gamma(-s_r) x_1^{s_1} \cdots x_r^{s_r} \, ds_1 \cdots ds_r \]

\[ (2.1) \]

where the contours are of Barnes type with indentations, if necessary, to ensure that the poles of \( \Gamma(A_j + s_1 + \cdots + s_r) \) are separated from those of \( \Gamma(-s_j), j = 1, \ldots, r \). The above result (1.23) can be easily established by an appeal to the calculus of residues by calculating the residues at the poles of \( \Gamma(-s_j), j = 1, \ldots, r \).

In order to evaluate a number of integrals of multivariable I-function, we first establish the formula

\[ \int_{a}^{b} (t - a)^{\alpha - 1} (b - t)^{\beta - 1} \prod_{j=1}^{l} \left[ 1 - \tau_j (t - a)^{h_j} \right]^{-\lambda_j} \prod_{j=1}^{k} (f_j t + g_j)^{\sigma_j} \, dt = (b - a)^{\alpha + \beta - 1} B(\alpha, \beta) \prod_{j=1}^{k} (af_j + g_j)^{\sigma_j} \]

\[ F_{1:0, \ldots, 0; 0, \ldots, 0}^{1:1, \ldots, 1} \left( \begin{array}{c} \alpha : h_1, \ldots, h_l, 1, \ldots, 1 \\ \lambda_1 : 1, \ldots, 1, \lambda_l : 1 \\ \sigma_1 : 1, \ldots, 1, \sigma_l : 1 \\ \alpha + \beta : h_1, \ldots, h_l, 1, \ldots, 1 \end{array} \right) = -, \ldots, -; -, \ldots, -; -, \ldots, - \]

\[ ; \tau_1 (b - a)^{h_1}, \ldots, \tau_l (b - a)^{h_l}, \frac{(b - a)f_1}{af_1 + g_1}, \ldots, \frac{(b - a)f_k}{af_k + g_k} \]

\[ (2.2) \]

where \( a, b \in \mathbb{R}(a < b), \alpha, \beta, f_j, g_j, \sigma_j, \tau_j, h_j, \lambda_j \in \mathbb{C}, \lambda_j \in \mathbb{R}^+ (i = 1, \ldots, k; j = 1, \ldots, l) \)

\[ \min(\text{Re}(\alpha), \text{Re}(\beta)) > 0, \max_{1 \leq j \leq l} \left\{ |\tau_j (b - a)^{h_j}| \right\} < 1, \max_{1 \leq j \leq k} \left\{ \left| \frac{(b - a)f_j}{af_j + g_j} \right| \right\} < 1, \]

and \( F_{1:0, \ldots, 0; 0, \ldots, 0}^{1:1, \ldots, 1} \) is a particular case of the generalized Lauricella function introduced by Srivastava-Daoust[4, page 454] and [5] given by
Here the contour are defined by starting at the point and terminating at the point with and each of the remaining contour run from \( -\omega \infty \) to \( \omega \infty \).

(2.2) can be easily established by expanding \( \prod_{j=1}^{l} [1 - \tau_j(t-a)^{h_j}]^{-\lambda_j} \) by means of the formula:

\[
(1-z)^{-\alpha} = \sum_{r=0}^{\infty} \frac{(\alpha)_r}{r!} z^r \quad (|z| < 1)
\]

integrating term by term with the help of the integral given by Saigo and Saxena [2, page 93, eq.(3.2)] and applying the definition of the generalized Lauricella function [4, page 454].

3. Eulerian integral

In this section, we note:

\[
\begin{align*}
\theta_i &= \prod_{j=1}^{l} [1 - \tau_j(t-a)^{h_j}]^{-\zeta_j(i)} \quad \zeta_j(i) > 0 (i = 1, \ldots, r) \quad \theta_i' &= \prod_{j=1}^{l} [1 - \tau_j(t-a)^{h_j}]^{-\zeta_j'(i)} \quad \zeta_j'(i) > 0 (i = 1, \ldots, s) \\
\theta_i'' &= \prod_{j=1}^{l} [1 - \tau_j(t-a)^{h_j}]^{-\zeta_j''(i)} \quad \zeta_j''(i) > 0 (i = 1, \ldots, u) \\
K_1 &= (1 - \alpha - \sum_{i=1}^{v} \eta_{G_i, g_i} (\mu_i + \mu'_i); \mu_1, \ldots, \mu_s, \mu'_1, \ldots, \mu'_u, 1, \ldots, 1, \nu_1, \ldots, \nu_l) \\
K_2 &= (1 - \beta - \sum_{i=1}^{v} \eta_{G_i, g_i} (\rho_i + \rho'_i); \rho_1, \ldots, \rho_s, \rho'_1, \ldots, \rho'_u, 0, \ldots, 0, \tau_1, \ldots, \tau_l) \\
K_P &= [1 - A_j; 0, \ldots, 0, 0, \ldots, 0, 0, \ldots, 0, 1, \ldots, 1, p]
\end{align*}
\]
where for are defined respectively by (1.2) and (1.3)

We have the general Eulerian integral

\[
\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}\sum_{j=1}^k (f_j(t)+g_j)^{\gamma_j} \left( \begin{array}{c}
z_{1}(t-a)^{\mu_1}(b-t)^{\rho_1} \prod_{j=1}^k (f_jt+g_j)^{-\lambda_j^{(1)}} \\
\vdots \\
z_{\gamma}(t-a)^{\mu_\gamma}(b-t)^{\rho_\gamma} \prod_{j=1}^k (f_jt+g_j)^{-\lambda_j^{(\gamma)}}
\end{array} \right) dt
\]

\[
\times \begin{pmatrix}
\left( z_1(t-a)^{\mu_1}(b-t)^{\rho_1} \prod_{j=1}^k (f_jt+g_j)^{-\lambda_j^{(1)}} \right) \\
\vdots \\
\left( z_{\gamma}(t-a)^{\mu_\gamma}(b-t)^{\rho_\gamma} \prod_{j=1}^k (f_jt+g_j)^{-\lambda_j^{(\gamma)}} \right)
\end{pmatrix}
\]

\[
= z_1(t-a)^{\mu_1}(b-t)^{\rho_1} \prod_{j=1}^k (f_jt+g_j)^{-\lambda_j^{(1)}} \\
\vdots \\
z_{\gamma}(t-a)^{\mu_\gamma}(b-t)^{\rho_\gamma} \prod_{j=1}^k (f_jt+g_j)^{-\lambda_j^{(\gamma)}}
\]

\[
= \left( z_1(t-a)^{\mu_1}(b-t)^{\rho_1} \prod_{j=1}^k (f_jt+g_j)^{-\lambda_j^{(1)}} \right) \\
\vdots \\
= \left( z_{\gamma}(t-a)^{\mu_\gamma}(b-t)^{\rho_\gamma} \prod_{j=1}^k (f_jt+g_j)^{-\lambda_j^{(\gamma)}} \right)
\]

\[
= \begin{pmatrix}
z_1(t-a)^{\mu_1}(b-t)^{\rho_1} \prod_{j=1}^k (f_jt+g_j)^{-\lambda_j^{(1)}} \\
\vdots \\
z_{\gamma}(t-a)^{\mu_\gamma}(b-t)^{\rho_\gamma} \prod_{j=1}^k (f_jt+g_j)^{-\lambda_j^{(\gamma)}}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
z_1(t-a)^{\mu_1}(b-t)^{\rho_1} \prod_{j=1}^k (f_jt+g_j)^{-\lambda_j^{(1)}} \\
\vdots \\
z_{\gamma}(t-a)^{\mu_\gamma}(b-t)^{\rho_\gamma} \prod_{j=1}^k (f_jt+g_j)^{-\lambda_j^{(\gamma)}}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
z_1(t-a)^{\mu_1}(b-t)^{\rho_1} \prod_{j=1}^k (f_jt+g_j)^{-\lambda_j^{(1)}} \\
\vdots \\
z_{\gamma}(t-a)^{\mu_\gamma}(b-t)^{\rho_\gamma} \prod_{j=1}^k (f_jt+g_j)^{-\lambda_j^{(\gamma)}}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
z_1(t-a)^{\mu_1}(b-t)^{\rho_1} \prod_{j=1}^k (f_jt+g_j)^{-\lambda_j^{(1)}} \\
\vdots \\
z_{\gamma}(t-a)^{\mu_\gamma}(b-t)^{\rho_\gamma} \prod_{j=1}^k (f_jt+g_j)^{-\lambda_j^{(\gamma)}}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
z_1(t-a)^{\mu_1}(b-t)^{\rho_1} \prod_{j=1}^k (f_jt+g_j)^{-\lambda_j^{(1)}} \\
\vdots \\
z_{\gamma}(t-a)^{\mu_\gamma}(b-t)^{\rho_\gamma} \prod_{j=1}^k (f_jt+g_j)^{-\lambda_j^{(\gamma)}}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
z_1(t-a)^{\mu_1}(b-t)^{\rho_1} \prod_{j=1}^k (f_jt+g_j)^{-\lambda_j^{(1)}} \\
\vdots \\
z_{\gamma}(t-a)^{\mu_\gamma}(b-t)^{\rho_\gamma} \prod_{j=1}^k (f_jt+g_j)^{-\lambda_j^{(\gamma)}}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
z_1(t-a)^{\mu_1}(b-t)^{\rho_1} \prod_{j=1}^k (f_jt+g_j)^{-\lambda_j^{(1)}} \\
\vdots \\
z_{\gamma}(t-a)^{\mu_\gamma}(b-t)^{\rho_\gamma} \prod_{j=1}^k (f_jt+g_j)^{-\lambda_j^{(\gamma)}}
\end{pmatrix}
\]
Provided that

(A) \( a, b \in \mathbb{R} (a < b) ; \mu_i, \rho_i, \mu'_i, \rho'_i, \lambda_i, \lambda'_i \in \mathbb{R}^+ \), \( f_i, g_i, r_j, s_j \in \mathbb{C} \) (\( i = 1, \ldots , r; j = 1, \ldots ; s, v = 1, \ldots , k \))

(B) See I

(C) \( \max_{1 \leq j \leq k} \left\{ \left| \frac{(b - a)f_i}{af_i + g_i} \right| \right\} < 1 \)
(D) \[ \text{Re} \left[ \alpha + \sum_{i=1}^{s} \mu_i \min_{1 \leq j \leq m_i} \frac{d_j^{(i)}}{\delta_j^{(i)}} + \sum_{i=1}^{u} \mu'_i \min_{1 \leq j \leq m'_i} \frac{b_j^{(i)}}{\beta_j^{(i)}} \right] > 0 \]

\[ \text{Re} \left[ \beta + \sum_{i=1}^{s} \rho_i \min_{1 \leq j \leq m_i} \frac{d_j^{(i)}}{\delta_j^{(i)}} + \sum_{i=1}^{u} \rho'_i \min_{1 \leq j \leq m'_i} \frac{b_j^{(i)}}{\beta_j^{(i)}} \right] > 0 \]

(E) \[ U^{(k)} = \sum_{j=1}^{n} \alpha_j^{(k)} + \tau_i \sum_{j=n+1}^{p_i} \alpha_j^{(k)} + \sum_{j=1}^{n_k} \gamma_{j}(k) + \tau_i \sum_{j=n_k+1}^{p_i(k)} \gamma_{j}(k) - \tau_i \sum_{j=1}^{q_i} \beta_j^{(k)} - \sum_{j=1}^{m_k} \delta_j^{(k)} \]

\[ -\tau_i^{(k)} \sum_{j=m_k+1}^{q_i(k)} \delta_j^{(k)} \leq 0 \]

\[ U^{(k)} = \sum_{j=1}^{n'} \mu_j^{(k)} + \tau_i \sum_{j=n'+1}^{p_i'} \mu_j^{(k)} + \sum_{j=1}^{n_k'} \alpha_j^{(k)} + \tau_i \sum_{j=n_k'+1}^{p_i'(k)} \alpha_j^{(k)} - \tau_i \sum_{j=1}^{q_i'} \alpha_j^{(k)} - \sum_{j=1}^{m_k'} \beta_j^{(k)} \]

\[ -\tau_i^{(k)} \sum_{j=m_k'+1}^{q_i'(k)} \beta_j^{(k)} \leq 0 \]

(F) \[ A_i^{(k)} = \sum_{j=1}^{n} \alpha_j^{(k)} - \tau_i \sum_{j=n+1}^{p_i} \alpha_j^{(k)} - \tau_i \sum_{j=1}^{p_i(k)} \alpha_j^{(k)} - \tau_i \sum_{j=1}^{q_i} \gamma_j^{(k)} - \sum_{j=n_k+1}^{p_i(k)} \gamma_j^{(k)} \]

\[ + \sum_{j=1}^{m_k} \delta_j^{(k)} - \tau_i^{(k)} \sum_{j=m_k+1}^{q_i(k)} \delta_j^{(k)} > 0 \]

\[ B_i^{(k)} = \sum_{j=1}^{n'} \mu_j^{(k)} - \tau_i \sum_{j=n'+1}^{p_i'} \mu_j^{(k)} - \tau_i \sum_{j=1}^{p_i'(k)} \mu_j^{(k)} - \tau_i \sum_{j=1}^{q_i'} \alpha_j^{(k)} - \tau_i \sum_{j=n_k'+1}^{p_i'(k)} \alpha_j^{(k)} \]

\[ + \sum_{j=1}^{m_k'} \beta_j^{(k)} - \tau_i^{(k)} \sum_{j=m_k'+1}^{q_i'(k)} \beta_j^{(k)} > 0 \]

(G) \[ \left| \frac{1}{n} \prod_{j=1}^{n} (p_{j} t + q_{j})^{-\lambda_j^{(i)}} \right| \leq \frac{1}{2} A_i^{(k)} \pi, A_i^{(k)} \text{ is defined by } (1.12) \text{ and } \]

\[ \left| \frac{1}{n} \prod_{j=1}^{n} (p_{j} t + q_{j})^{-\lambda_j^{(i)}} \right| \leq \frac{1}{2} B_i^{(k)} \pi, B_i^{(k)} \text{ is defined by } (1.24) \]

(H) \[ P \leq Q + 1. \text{ The equality holds, when, in addition,} \]
either $P > Q$ and
\[
\left| \sum_{i=1}^{k} \left( \prod_{j=1}^{k} (f_j t + g_j) - c_j^{(i)} \right) \right| < 1 \quad (a \leq t \leq b)
\]
or $P \leq Q$ and
\[
\max_{1 \leq i \leq k} \left| \left( \sum_{i=1}^{k} \prod_{j=1}^{k} (f_j t + g_j) - c_j^{(i)} \right) \right| < 1 \quad (a \leq t \leq b)
\]

(1) The multiple series occuring on the right-hand side of (3.14) is absolutely and uniformly convergent.

**Proof**

First expressing the the multivariable A-function in serie with the help of (1.1) and we interchange the order of summations and t-integral (which is permissible under the conditions stated). Expressing the Aleph-function of s-variables and u-variables by the Mellin-Barnes contour integral with the help of the equation (1.8) and (1.20) respectively, the generalized hypergeometric function $pF_q$ in Mellin-Barnes contour integral with the help of (2.1) and interchange the order of integrations which is justifiable due to absolute convergence of the integral involved in the process. Now collect the power of $(f_j t + g_j)$ with $j = 1, \cdots, k$ and use the equations (2.1) and (2.2) and we obtain $k$–Mellin-Barnes contour integral. Changing the order of integration and summation (which is easily seen to be justified due to the absolute convergence of the integral and the summations involved in the process) and interpreting $(r + s + k + l)$–Mellin-barnes contour integral to multivariable Aleph-function, we obtain the desired result.

4. Particular case

If $\tau, \tau_{(r)}, \cdots, \tau_{(s)}, \tau_{(r)}, \tau_{(s)}, \cdots, \tau_{(u)} \rightarrow 1$, the Aleph-function of r-variables and the Aleph-function of s-variables reduces respectively to I-function of r-variables and I-function of s-variables defined by Sharma et al [3],and we have :

We have the general Eulerian integral

\[
\int_{a}^{b} \frac{\prod_{j=1}^{k} (f_j t + g_j)^{\sigma_j} A}{(t-a)^{\alpha-1}(b-t)^{\beta-1}} \left( z_{1}^{(1)}(t-a)^{\mu_1} \prod_{j=1}^{k} (f_j t + g_j)^{-\lambda_j^{(1)}} \right)
\]

\[
\vdots
\]

\[
\left( z_{v}^{(v)}(t-a)^{\mu_v} \prod_{j=1}^{k} (f_j t + g_j)^{-\lambda_j^{(v)}} \right)
\]

\[
I \left( z_{1}^{(1)}(t-a)^{\mu_1} \prod_{j=1}^{k} (f_j t + g_j)^{-\lambda_j^{(1)}} \right)
\]

\[
\vdots
\]

\[
I \left( z_{v}^{(v)}(t-a)^{\mu_v} \prod_{j=1}^{k} (f_j t + g_j)^{-\lambda_j^{(v)}} \right)
\]

\[
pFq \left( (A_p); (B_q); - \sum_{i=1}^{l} \frac{t^{(v_i)}}{(b-t)^{v_i}} \prod_{j=1}^{k} (f_j t + g_j)^{-\zeta_j^{(i)}} \right) dt = (b-a)^{\alpha+\beta-1}
\]
under the same notations and conditions that (3,14) with \( \tau, \tau(1), \ldots, \tau(r), \ell, \ell(1), \cdot \ldots, \ell(u) \to 1 \)

**Remark**

By the following similar procedure, the results of this document can be extented to product of any finite number of multivariable Aleph-functions.

5. Conclusion

In this paper we have evaluated a generalized Eulerian integral involving the product of two multivariable Aleph-functions, a expansion of a multivariable A-function and a generalized hypergeometric function with general arguments. The formulae established in this paper is very general nature. Thus, the results established in this research work would serve as a key formula from which, upon specializing the parameters, as many as desired results involving the special functions of one and several variables can be obtained.

**REFERENCES**

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