Quasi-static Von-Karman evolution and Numerical approach

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Abstract

In this paper we consider the vibration of nonlinear deformation of elastic shallow shell. This is a parabolic problem of Von-Karman evolution without rotational inertia, in quasi-static form. The aim of this article is to finding a condition veriﬁed by the external and internal loads and the linear bounded operator in up to have a weakly uniqueness solution of the Von-Karman evolutions, without rotational inertia with clamped boundary conditions, in quasi-static form, by another approach distinct from the preceding presented in [3], one which yields immediately a simple numerical approach of ﬁnite difference method to the considered problem. This paper is organized as follows. In section 2 we present the some theoretical results for established a uniqueness weak solution. But in the third section we use the noncoupled approach of 13-point and the alternating direction implicit schemes (ADI).

Keywords: Elastic shallow shell, Quasi-static Von-Karman, Finite difference method, ADI methods.

1. Introduction

In [3] I.Chueshov and I.Lasiecka present the problem of quasi-static Von-Karman evolution and establish the existence of weak solution, but the uniqueness is not known. However the existence and uniqueness dose hold for strong solution. This model in the quasi-static form of clamped boundary conditions describe the case when the inertia forces are small in comparison with the resisting forces of the medium (µ = 0). We well know the quasi-static Von-Karman evolution without rotational inertia (α = 0), for vertical displacement u(x, y, t) and the Airy stress φ(x, y, t) has the following form [3] :

\[
\begin{align*}
(P_0) & \left\{ \begin{array}{ll}
u_t + \Delta^2 u - [\phi + F_0, u + \theta] + L(u) = p & \text{in } \omega \times [0, T], \\
u_{t_{|t=0}} = u_0 & \text{in } \omega, \\
\Delta \phi + |u, u + 2\theta| = 0 & \text{on } \Gamma \times [0, T], \\
\phi = 0, \partial_{\nu} \phi = 0 & \text{on } \Gamma \times [0, T].
\end{array} \right.
\end{align*}
\]

Where ω is the middle surface of the shell, u0 is initial datum and [...] is the Monge-Ampère operator [10]. The shell is subjected to the internal force F0 and external force p, but θ(x, y) [2] is the mapping measuring the deviation of the middle surface of the reference configuration of the shell from a plane, moreover L is a linear operator source and characterize the non conservative potentially loads to the system.

The aim of this article is to ﬁnd a condition veriﬁed by the external and internal loads and the linear bounded operator in up to have a weakly uniqueness solution of the Von-Karman evolutions, without rotational inertia with clamped boundary conditions, in quasi-static form, by another approach distinct from the preceding presented in [3], one which yields immediately a simple numerical approach of ﬁnite difference method to the considered problem.

This paper is organized as follows. In section 2 we present the some theoretical results for established a uniqueness weak solution. But in the third section we use the noncoupled approach of 13-point and the alternating direction implicit schemes [11] for approached this solution.

2. Dynamic quasi-static Von-Karman equations

In this paper, ω denotes a nonempty bounded open domain in \( IR^2 \), with regular boundary \( \Gamma = \partial \omega \).

Let us consider the following problem:[3]

\[
\text{Fund } (u, \phi) \in (L^2([0, T], H^2_0(\omega)))^2 \text{ such that }
\]

\[
\begin{align*}
(P_0) & \left\{ \begin{array}{ll}
u_t + \Delta^2 u - [\phi + F_0, u + \theta] + L(u) = p & \text{in } \omega \times [0, T], \\
u_{t_{|t=0}} = u_0 & \text{in } \omega, \\
\Delta \phi + |u, u + 2\theta| = 0 & \text{on } \Gamma \times [0, T], \\
\phi = 0, \partial_{\nu} \phi = 0 & \text{on } \Gamma \times [0, T].
\end{array} \right.
\end{align*}
\]

Where \( T > 0 \) is a real number, \( u_t = \partial_{tt} \) and

\[
[\phi, u] = \partial_{11} \partial_{22} u + \partial_{12} \partial_{21} u + 2\partial_{13} \partial_{23} \phi - 2\partial_{23} \partial_{13} u.
\]

Let \( p \geq 1 \) and \( m \in \mathbb{N}^* \), we put by:

\[
|u|_{p} = \left( \int_{\omega} |u|^p \right)^{1/p}, \quad \|u\|_{m, \omega} = \|\Delta u\|_{2}, \quad \|u\|_{m, \omega} \text{ the classical norm in } H^m(\omega) \text{ and }
\]

\[
W(0, T) = \{ u / u \in L^2([0, T], H^2_0(\omega)), u_t \in L^2([0, T], L^2(\omega)) \} \text{ is a complete Hilbert space with associated norm }
\]

\[
\|\cdot\|_{W(0, T)} = (\|\cdot\|_{L^2([0, T], L^2(\omega))}^2 + \|\cdot\|_{L^2([0, T], H^2_0(\omega))}^2)^{1/2}.
\]
Theorem 2.1 [7, 8] Let \( f \in L^2(\omega) \), then the next problem \((Q)\):
\[
\begin{align*}
\Delta^2 v &= f \quad \text{in} \quad \omega, \\
v &= 0 \quad \text{on} \quad \Gamma, \\
\partial_n v &= 0 \quad \text{on} \quad \Gamma.
\end{align*}
\]

Has one and only one solution \( v \) in \( H_0^2(\omega) \cap H^4(\omega) \) satisfying that \( \|v\| \leq c_0 \|f\|_2 \). Where \( c_0 > 0 \) is a constant which depends only of \( m(\omega) \).

Remark 2.1 If \( f \in L^2([0, T], L^2(\omega)) \), the uniqueness solution of the problem \((Q)\) is in \( L^2([0, T], H_0^2(\omega) \cap H^4(\omega)) \).

Theorem 2.2 [6] Let \( 0 < T \leq +\infty \) and \( f \in L^2([0, T], L^2(\omega)) \). The Dirichlet problem for linear fourth order parabolic equation:
\[
u + \Delta^2 u = f \quad \text{in} \quad \omega \times [0, T]
\]
with initial datum \( u_0 \in H_0^2(\omega) \) admits a unique weak solution in the space \( C([0, T], H^2(\omega)) \cap L^2([0, T], H^4(\omega)) \cap H^1([0, T], L^2(\omega)) \).

The corresponding problem with initial datum \( u_0 \) in \( H_0^2(\omega) \cap H^2(\omega) \) admits a unique weak solution in the following space \( C([0, T], H^2(\omega) \cap H^4(\omega)) \cap L^2([0, T], H^4(\omega)) \cap H^1([0, T], L^2(\omega)) \). Furthermore, both cases admit the estimate.
\[
\sup_{0 \leq t \leq T} \|u\|^2 + \int_0^T \|u\|^2 + \int_0^T |u|^2 \leq c(\|u_0\|^2 + \int_0^T |f|^2).
\]

We will study the problem \((P_0)\) by considering the following iterative problem:

Let \( n \geq 2 \) and \( 0 \neq u^1(x, y) \in H_0^2(\omega) \) is given. In the firstly we find \( \phi_{n-1} \in H_0^2(\omega) \) as a solution to the problem \( \Delta^2 \phi_{n-1} = -[u_{n-1}, u_{n-1} + 2\theta] \) and \( u_n \in L^2([0, T], H_0^2(\omega)) \) is constructed by the following problem:
\[
(P_n) \begin{cases}
\Delta^2 u_n + \Delta u_n &= f_1(u_{n-1}, \phi_{n-1}) + p \quad \text{in} \quad \omega \times [0, T], \\
u_n &= \partial_n u_n = 0 \quad \text{on} \quad \Gamma \times [0, T], \\
(u_n, \phi_{n-1}) &= u_0 \quad \text{in} \quad \omega.
\end{cases}
\]

Where \( F(u_{n-1}, \phi_{n-1}) = (F_1, F_2) = ([\phi_{n-1} + F_0, u_{n-1} + \theta] - L(u_{n-1}), -[u_{n-1}, u_{n-1} + 2\theta]) \).

Remark 2.2 By virtues of the theorem 2.1 and theorem 2.2. If \( \forall n \geq 0, (u_n, \phi_{n-1}) \) is a solution of the problem \((P_n)\), this solution has the regularity:
\[
(u_n, \phi_{n-1}) \in L^2([0, T], H^4(\omega) \cap H^2(\omega)) \times H^4(\omega) \cap H^2(\omega).
\]

Theorem 2.3 Let \( c > 0, \tilde{u} = (u, \psi) \) and \( \tilde{v} = (v, \varphi) \) in \( L^2([0, T], H^4(\omega) \cap H^2(\omega)) \times H^4(\omega) \cap H^2(\omega) \) such that \( \sup_{0 \leq t \leq T} \|\tilde{u}\|_{H_0^2(\omega) \cap H^2(\omega)} \leq c \) and \( \|\tilde{v}\|_{H_0^2(\omega) \cap H^2(\omega)} \leq c \). If \( \|\theta\|_{L^2(\omega)} \leq 1 \) and \( \|\tilde{v}\|_{L^2(\omega)} \leq 1 \), then there exists \( 0 < c_1 < 1 \) such that
\[
\left\| \tilde{F} - \tilde{F} \right\|_{(L^2(\omega))^2} \leq c_1 \|\tilde{u} - \tilde{v}\|_{H_0^2(\omega) \cap H^2(\omega)}.
\]

Proof Let \( \tilde{u} = (u, \psi) \) and \( \tilde{v} = (v, \varphi) \) in \( L^2([0, T], H^4(\omega) \cap H^2(\omega)) \times H^4(\omega) \cap H^2(\omega) \) such that \( \|\tilde{u}\|_{H_0^2(\omega) \cap H^2(\omega)} \leq c \) and \( \|\tilde{v}\|_{H_0^2(\omega) \cap H^2(\omega)} \leq c \), we have
\[
\|\tilde{F}(\tilde{u}) - \tilde{F}(\tilde{v})\|_{(L^2(\omega))^2}^2 \leq \|\psi - \varphi, \theta\| + \|u - v, F_0\|_{(L^2(\omega))^2}^2 + \|u - v, \theta\| + \|u - v, F_0\|_{(L^2(\omega))^2}^2.
\]

It follows that
\[
\|\psi - \varphi, \theta\| + \|u - v, F_0\|_{(L^2(\omega))^2}^2 \leq \|\psi - \varphi, \theta\| + \|u - v, F_0\|_{(L^2(\omega))^2}^2.
\]

Using the injection \( H^4(\omega) \hookrightarrow \mathbb{R}^2 \) we have
\[
\|\psi - \varphi, \theta\| + \|u - v, F_0\|_{(L^2(\omega))^2}^2 \leq \|\psi - \varphi, \theta\| + \|u - v, F_0\|_{(L^2(\omega))^2}^2.
\]

By an analogous method we have
\[
\|\psi - \varphi, \theta\| + \|u - v, F_0\|_{(L^2(\omega))^2}^2 \leq \|\psi - \varphi, \theta\| + \|u - v, F_0\|_{(L^2(\omega))^2}^2.
\]

Remark 2.3 By virtue of the theorem 2.3 there exist a constant \( 0 < c_1 < 1 \) such that
\[
|F_1(u) - F_1(v)| \leq \left\| F_1(u) - F_1(v) \right\|_{(L^2(\omega))^2} \leq c_1 \|u - v\|_{(L^2(\omega))^2}^2.
\]

And by analogous method from the theorem 2.3 we can proved that. \( |F_2(u) - F_2(v)| \leq c_1 \|u - v\| \).

Proposition 2.1 Let \( u, v \) in \( H_0^2(\omega) \) and \( \theta \) in \( H^2(\omega) \). Of small norm. If \( \phi \) and \( \psi \) are two solutions of the following Dirichlet problem:
\[
\Delta^2 \phi = -[u + 2\theta, \psi] \quad \text{and} \quad \Delta^2 \psi = -[v + 2\theta, v].
\]

Then, there exist \( 0 < c_1 < 1 \), such that
\[
|\phi - \psi, \theta| \leq c_1 \|u - v\|.
\]

Proof Let \( c > 0, \|\phi\| \leq c \) and \( |v| \leq c \). In [3] we have
\[
|\phi - \psi, \theta| \leq c_0 \|\phi\|^2 + \|\theta\|^2_{L^2(\omega)} \|u - v\| \leq c_0 (2c^2 + \|\theta\|^2_{L^2(\omega)}) \|u - v\|.
\]

If we choose \( c \) sufficiently small, \( c_1 = 2c^2 + \|\theta\|^2_{L^2(\omega)} \leq 1 \) and \( 0 < c_1 \leq 1 \), we conclude that
\[
|\psi - \varphi, \theta| \leq c_1 \|u - v\| \quad \text{and} \quad 0 < c_1 < 1.
\]

Proposition 2.2 Let \( u \in H_0^2(\omega) \) and \( \phi \) be a uniqueness solution of the following problem:
\[
\Delta^2 \phi + [u, u + 2\theta] = 0 \quad \text{in} \quad \omega \times [0, T], \\
\frac{\partial \phi}{\partial n} = 0 \quad \text{on} \quad \Gamma \times [0, T].
\]

Then, there exist a constant \( K > 0 \) such that
\[
|\phi - F_0 - u + \theta| \leq K \|\phi\|^2 + K.
\]

Where \( \ell \) is the duality operator in \( L^2(\omega) \).
Proof: Let $u \in H^2_0(\omega)$, then 

$$\left\{ \phi + F_0, u + \theta \right\} L^2_0(\omega) \ni \left( \left\| u, u + 2\theta \right\|, \phi \right) L^2_0(\omega) - \left\{ \phi, u \right\} L^2_0(\omega) \ni \left( u, u + \theta \right), \theta \right\} L^2_0(\omega) \ni \left( \left\| u, u + \theta \right\|, F_0 \right) L^2_0(\omega).$$

By using the Green formula and the injection $H^2(\omega) \hookrightarrow C(\overline{\omega})$, we have 

$$\left\{ \phi + F_0, u + \theta \right\} L^2_0(\omega) \ni \left( \left\| u, u + \theta \right\|, F_0 \right) L^2_0(\omega).$$

According to the theorem 2.3, there exist a constant $K > 0$ which depend only of $\left\| \theta \right\|_{L^2_0(\omega)}$, $F_0 \in L^2_\omega(\omega)$, such that 

$$\left\{ \phi + F_0, u + \theta \right\} L^2_0(\omega) \ni \left( \left\| u, u + \theta \right\|, F_0 \right) L^2_0(\omega) \ni \left( \left\| u, u + \theta \right\|, F_0 \right) L^2_0(\omega).$$

Therefore we have $K > 0$ which depend only of $\left\| \theta \right\|_{L^2_0(\omega)}$ and $F_0 \in L^2_\omega(\omega)$. 

Theorem 2.4 Let $p(x, y) \in L^2(\omega)$ and $u_0 \in H^2_0(\omega)$. If 

$$\left\| f \right\|_{L^2_0(\omega)}, \left\| p \right\|_{L^2_0(\omega)}, \left\| u \right\|_{L^2(\omega)}$$

are small and $\left\| L \right\| < 1$, then the problem $(P_0)$ has one and only small solution $(u, \phi)$ in the following space: 

$$(C([0, T], H^2_0(\omega)) \cap L^2([0, T], H^2_0(\omega))) \ni \sum_{k=0}^{n-1} \left( \left\| u, u \right\|_{L^2_0(\omega)}, \left\| \phi \right\|_{L^2_0(\omega)} \right) \ni \left\{ \left\| u, u \right\|_{L^2_0(\omega)}, \left\| \phi \right\|_{L^2_0(\omega)} \right\} \ni \left( \left\| u, u \right\|_{L^2_0(\omega)}, \left\| \phi \right\|_{L^2_0(\omega)} \right).$$

If we choose $c > 0$ sufficiently small, then $0 < c_1 < 1$, 

$$0 < c_0 < 1, 0 < 8c_0^2 c_1 T < 1,$$

$$\left\| u_0 \right\|_{L^2_0(\omega)} < \frac{1}{8c_0^2 c_1 T} \left\| u \right\|_{L^2_0(\omega)},$$

and 

$$\left\| u \right\|_{L^2_0(\omega)} < \frac{1}{8c_0^2 c_1 T} \left\| u \right\|_{L^2_0(\omega)}.$$
Finally we deduce that the formula of finite difference developed by M. Gubta in [9].

First step: we find $v$.

Second step: We present the alternating direction scheme developed by T. P. Witelski and M. Bowen in [11], to the following parabolic problem:

\[
\begin{cases}
\Delta^2 v = f & \text{in } \omega \times [0, T], \\
\partial_{\nu} v = g_1 & \text{on } \Gamma, \\
\partial_{\nu} v = g_2 & \text{on } \Gamma.
\end{cases}
\]

3. Numerical application

In this section let $\omega$ be the square $[0,1] \times [0,1]$ in $IR^2$ and $T > 0$. For approached the weak uniqueness solution of the quasi-state Von-Karman evolution without rotational inertia, we utilize the following iterative method:

\[
\begin{cases}
\phi_n(x,y) & \text{is given in } H^2_0(\omega), \\
\phi_{n+1} = \phi_n - p_n v_n, & \text{with } v_n = \frac{1}{\Delta}_n v_{n-1} - \Delta \phi_n, \\
\phi_{n+1} = \phi_n - p_n v_n, & \text{with } v_n = \frac{1}{\Delta}_n v_{n-1} - \Delta \phi_n, \\
\phi_{n+1} = \phi_n - p_n v_n, & \text{with } v_n = \frac{1}{\Delta}_n v_{n-1} - \Delta \phi_n,
\end{cases}
\]

3.1. Noncoupled Approach

In [9], M. Gupta presents the numerical analysis of finite-difference method for solving the Biharmonic equation. This method has known that of noncoupled method of 13-point. Moreover Glowinski and Pironneau [7] made the observation that the 13-point finite difference scheme combined with a quadratic extrapolation formula near the boundary is equivalent to mixed finite element method with piecewise linear elements.

3.1.1. Discrete formulation of 13-point

In order to solve the problem (P) of Biharmonic equation numerically, we introduce a uniform mesh of width $h$. Let $\omega_h$ be the set of all mesh points inside $\omega$ with internal points $x_i = ih$, $y_j = jh$. $i, j = 1, \ldots, N - 1$, $h = \frac{1}{N+1}$. $\omega_h$ be the set of boundary mesh points and $v_h$ represent the finite-difference approximation of $v$.

Lemma 3.1. [9] The 13-point approximation of the Biharmonic equation for approximating the uniqueness solution $v$ of the problem (P) is defined by:

\[
L_h v_{ij} = h^{-4} (v_{ij-2} + v_{ij+2} + v_{i-2,j} + v_{i+2,j} - 8(v_{ij-1} + v_{ij+1}) + 2(v_{ij-1} + v_{ij+1} + v_{i+1,j} + v_{i-1,j} + v_{i+1,j-1} + v_{i+1,j+1} - v_{i-1,j+1}),
\]

where $v_{ij} = v(x_i, y_j)$.

When the mesh point $(x_i, y_j)$ is adjacent to the boundary $\partial \omega_h$, the mentioned values of $v_h$ are conventionally calculated by the following approximation of $\partial v$ defined by [9]:

\[
v_{i-1,j} = \frac{3}{2} v_{ij} - \frac{3}{2} v_{i+1,j} - h(\partial_x v)_{ij-1},
\]

\[
v_{i,j-1} = \frac{3}{2} v_{ij} - \frac{3}{2} v_{ij+1} - h(\partial_y v)_{ij-1},
\]

\[
v_{i+1,j} = \frac{3}{2} v_{ij} + \frac{3}{2} v_{ij-1} - h(\partial_x v)_{ij+1},
\]

\[
v_{i,j+1} = \frac{3}{2} v_{ij} + \frac{3}{2} v_{ij-1} - h(\partial_y v)_{ij+1},
\]

Remark 3.1. In [9], M. Gubta generalized the approximation of the $\partial v$ known that by the $(p,q)$ formula or the two-point formula. In the next Lemma 3.1 the approximation of $\partial v$ correspond at the $(2,0)$ formula.

3.2.1. Matrix system of scheme (1)

Let $V = (v_{11}, v_{12}, \ldots, v_{1N-1}, v_{21}, \ldots, v_{2N-1}, \ldots, v_{N-1N-1})$ be a vector of unknown values of the approached solution $v_h$, by using the 13-point finite difference method, the discretized problem:
is a matrix system of scheme (1) of order \((N-1)^2\) and \(F\) a known vector depend only of body forces \(f_1\) and lateral forces \(g_0, g_1\).

Such that

\[
L_h v_{ij} = n^{-1} [v_{ij-2} + v_{ij+2} + v_{ij-1} + v_{ij+1} - 8(v_{ij-1} + v_{ij+1}) + 8(v_{ij+1} + v_{ij+1}) + 2(v_{i-1j+1} + v_{i+1j+1} + v_{i+1j-1}) + 2v_{i+1j+1} - 20v_{ij}] = f_1(x_i, y_j), \text{ for } i,j=1,...,N-1
\]

is equivalent to the linear system \(AV = \bar{F}\), where \(A\) is a matrix system of scheme (1) of order \((N-1)^2\) and \(\bar{F}\) a known vector depend only of body forces \(f_1\) and lateral forces \(g_0, g_1\).

3.2. Numerical solution of parabolic problem

In [11], T.P.Witelski and M.Bowen present a new finite difference approximation to the last problem (**), known that of alternating direction implicit schemes (ADI) and study the stability and convergence. Moreover the authors generalize the some results of the (ADI) scheme in the case of linear problem to the nonlinear equations.

3.2.1. Discrete formulation of finite difference method

In order to solve the problem (**) of parabolic equation numerically, we introduce a uniform mesh presented in the last subsection 3.1.1 and we introduce the next typical notation for difference operators :

\[
w(i\Delta x, j\Delta y, n\Delta t) = w_{ijn}
\]

\[
\Delta_t w_{ijn} = \frac{\partial w_{ijn}}{\partial t} = \frac{w_{ijn+1} - w_{ijn}}{(\Delta t)}
\]

\[
\Delta_x w_{ijn} = \frac{\partial^2 w_{ijn}}{\partial x^2} = \frac{w_{ij+1n} - 2w_{ijn} + w_{ij-1n}}{(\Delta x)^2}.
\]

Now we approximate the problem (**), by the following finite difference (ADI) system presented by T.P.Witelski and M.Bowen in [11] :

\[
\left\{ \begin{array}{ll}
L_x w^* = - (\Delta t) \Delta_x^2 w_{ijn} + f_{ijn} \\
L_y v^* = w^*
\end{array} \right.
\]

\[
w_{ijn+1} = w_{ijn} + v^* \text{ for } ij = 1, ..., N - 1
\]

**Boundary conditions**

\[
w_{0jn} = w_{Njn} = w_{0nn} = w_{NNn} = 0 \text{ for } i = 0, ..., N,
\]

\[
 j = 0, ..., N \text{ and } 0 \leq n \Delta t \leq T
\]

\[
w_{ij0} = (w_0)_{ij0} \text{ for } ij = 0, ..., N
\]

Where \(w^*\) and \(v^*\) represent an intermediate results obtained from solving the first and second equations, but \(L_x = I + \theta(\Delta t)\Delta_x^2, L_y = I + \theta(\Delta t)\Delta_y^2\) are two operators and \(0 \leq \theta \leq 1\).

3.2.2. Matrix system of scheme (2)

Let \(W^n = (w_{11n}, w_{12n}, ..., w_{1N-1n}, w_{21n}, ..., w_{2N-1n}, ..., w_{N-11n}, w_{N-12n}, ..., w_{NNn})\) be a vector of unknown values of the approached weakly uniqueness solution \(w_h\) of the problem (**), by using the next (ADI) scheme (2) of finite difference approximation to the parabolic problem :

\[
\left\{ \begin{array}{ll}
L_x w^* = - (\Delta t) \Delta_x^2 w_{ijn} + f_{ijn} \\
L_y v^* = w^*
\end{array} \right.
\]

\[
w_{ijn+1} = w_{ijn} + v^* \text{ for } ij = 1, ..., N - 1
\]

This scheme (2) presented under matrix form, is equivalent to the following linear system :

\[
BW^* = AW^n + F^n,
\]

\[
CV^* = W^*,
\]

\[
W^{n+1} = W^n + V^*.
\]

Where \(A, B = I + \theta(\Delta t)B_2\) and \(C = I + \theta(\Delta t)C_1\) are tree matrix of order \((N-1)^2\) and \(F\) is the known vector depend only of body forces and the boundary conditions. Such that

Theorem 3.1 [9] The scheme (1) of 13-point is convergent and the error is of \(h^2\) order.
Where $C_1 = (C_{ij}^1)_{1 \leq ij \leq N - 1}$ is matrix of block diagonal matrix and $A$ is the matrix of scheme (1).

**Example 1.** We consider in this example the following analytical external forces $p$, internal forces $F_0$, source $L$ and the mapping $\theta$.

$$F_0(x, y) = 0.8xe^{-x^2-y^2}$$
$$L(u) = 0.8x(e^{-x^2} - e^{-y^2})u$$
$$\theta(x, y) = -0.5xy(x - 1)(y - 1)e^{-x^2-y^2}$$
$$p(x, y) = \sin^2(\pi x) \cos^2(\pi x)$$

**References**


