Fixed Point Theorems for Self-Mappings in a Menger Space using Contractive Control Function under CLR / JCLR-Property

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Abstract. In this paper, using contractive control function and CLR / JCLR property two fixed point theorems for self-mappings in a Menger space are mainly proved and also considered a variant of those theorems. Examples are provided in support of the theorems.

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1. INTRODUCTION

K. P. R. Sastry et. al[2] proved a fixed point theorem for four self-mappings on a complete Menger space using compatible, weakly compatible and continuity concepts. Now, this result is extended to six self-mappings and proved under weaker conditions using CLR / JCLR property.

Throughout this paper $\mathbb{R}$ and $\mathbb{R}^+$ stand for the set of all reals and the set of all non-negative reals respectively.

2. PRELIMINARIES AND BASIC RESULTS

Definition 2.1. ([3]) A function $F : \mathbb{R} \to \mathbb{R}^+$ is said to be a distribution function if and only if

(i) $F$ is non-decreasing (i.e. monotonic increasing),
(ii) $F$ is left continuous and
(iii) $\inf\{F(u) : u \in \mathbb{R}^+\} = 0$ and $\sup\{F(u) : u \in \mathbb{R}^+\} = 1$.

$D$ denotes the family of all distribution functions on $\mathbb{R}^+$. $H$ is a special element of $D$ defined by

$$H(u) = \begin{cases} 0 & \text{if } u \leq 0 \\ 1 & \text{if } u > 0. \end{cases}$$

($H$ is called the Heaviside function.)

Definition 2.2. ([3]) Let $X$ be a non-empty set and $D$ denotes the set of all distribution functions. The ordered pair $(X, F)$ is called a probabilistic metric space if and only if $F$ is a mapping from $X \times X$ into $D$ satisfying the following conditions:

(i) $F_{x,y}(u) = H(u)$ if and only if $y = x$;
(ii) $F_{x,y}(u) = F_{y,x}(u)$;
(iii) $F_{x,y}(0) = 0$, and
(iv) if $F_{x,z}(u) = 1$, $F_{z,y}(v) = 1$ then $F_{x,y}(u + v) = 1$

for all $x, y, z$ in $X$ and $u, v > 0$.

($F_{x,y}$ is the distribution function $F(x, y)$ associated with $(x, y)$).

Every metric space $(X, d)$ can be viewed as a probabilistic metric space by taking $F_{x,y}(u) = H(u - d(x, y))$ for all $x, y$ in $X$.

Definition 2.3. ([3]) A mapping $*: [0, 1] \times [0, 1] - [0, 1]$ is said to be a triangular norm (known as t-norm) if and only if $*$ satisfies the following conditions. For all $a, b, c$ in $[0, 1]$,
(i) \(*(a, 1) = a\) and \(*(0, 0) = 0\);
(ii) \(*(a, b) = *(b, a)\);
(iii) \(*(a, b) \leq *(c, d)\) whenever one of \(a, b\) is \(\leq c\) and the other is \(\leq d\), and
(iv) \(**(a, b, c) = *(a, \ast(b, c))\).

\(\ast(a, b)\) is denoted by \(a \ast b\).

If further \(\ast\) is continuous on \([0, 1] \times [0, 1]\) (under the usual metric) then it is called a continuous triangular norm.

**Definition 2.4.** ([3]) A Menger space is an ordered triad \((X, F, \ast)\) where \(\ast\) is a triangular norm and \((X, F)\) is a probabilistic metric space satisfying the following condition:

\[
F_{x,z}(u + v) \geq \ast(F_{x,y}(u), F_{y,z}(v)) = F_{x,y}(u) \ast F_{y,z}(v)
\]

for all \(x, y, z \in X\) and \(u, v > 0\).

**Definition 2.5.** ([4]) Self maps \(A\) and \(B\) of a Menger space \((X, F, \ast)\) are said to be weakly compatible if and only if they commute at their coincidence points; i.e., if \(Ax = Bx\) for some \(x \in X\) then \(ABx = BAx\).

**Definition 2.6.** ([2]) A mapping \(\zeta : \mathbb{I}^+ \rightarrow \mathbb{I}^+\) is such that \(\zeta\) is strictly increasing and for some \(\alpha \in (1, 2), (\alpha - 1)\zeta(t) > t\) for all \(t > 0\), then \(\zeta\) is called a contractive control function.

**Notation:** \(Z\) stands for the class of all contractive control functions.

**Definition 2.7.** ([11]) Let \((X, F, \ast)\) be a Menger space and \(A, S\) be self mappings on \(X\). The pairs \(\{A, S\}\) and \(\{B, T\}\) share common property \((E.A)\) if and only if there exist sequences \(\{x_n\}\) and \(\{y_n\}\) and a point \(z \in X\) such that

\[
\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} By_n = \lim_{n \to \infty} Ty_n = z.
\]

**Definition 2.8.** ([5]) Let \((X, F, \ast)\) be a Menger space, where \(\ast\) denotes a continuous \(t\)-norm and \(f, g, h, k\) be self mappings on \(X\). The pairs \(\{f, g\}\) and \(\{h, k\}\) are said to satisfy the "common limit in the range of \(g\)" (CLR\(_g\)) - property if and only if there exist sequences \(\{x_n\}\) and \(\{y_n\}\) and a point \(u \in X\) such that

\[
\lim_{n \to \infty} F_{x_n, gu}(t) = \lim_{n \to \infty} F_{x_n, gu}(t) = \lim_{n \to \infty} F_{y_n, gu}(t) = \lim_{n \to \infty} F_{y_n, gu}(t) = 1, \text{ for all } t > 0.
\]

Similar is the case with CLR\(_f\), CLR\(_h\), CLR\(_k\)-properties where \(gu\) is replaced by \(fu, hu, ku\) in the above equality quantities.

**Definition 2.9.** ([6]) Let \((X, F, \ast)\) be a Menger space, where \(\ast\) denotes a continuous \(t\)-norm and \(f, g, h, k\) be self mappings on \(X\). The pairs \(\{f, g\}\) and \(\{h, k\}\) are said to satisfy the "joint common limit in the range of \(g\)" (JCLR\(_g\)) - property if and only if the exist sequences \(\{x_n\}\) and \(\{y_n\}\) and a point \(u \in X\) such that

\[
\lim_{n \to \infty} F_{x_n, gu}(t) = \lim_{n \to \infty} F_{x_n, gu}(t) = \lim_{n \to \infty} F_{y_n, gu}(t) = \lim_{n \to \infty} F_{y_n, gu}(t) = 1, \text{ for all } t > 0.
\]

Similar is the case with respect to other relevant combinations.

### 3. MAIN RESULTS

K. P. R. Sastry et. al[2] proved the following:

**Result 3.1.** Let \(A, B, S\) and \(T\) be self mappings on a complete Menger space \((X, F, \ast)\) where \(\ast\) is the min \(t\)-norm and satisfying:

(i) \(A(X) \subseteq T(X)\) and \(B(X) \subseteq S(X)\);
(ii) \(F_{Ax,Ty}(t) \geq F_{ASty}(\zeta(t)) \ast F_{AxSx}(\zeta(t)) \ast F_{By,Ty}(\zeta(t)) \ast F_{BySy}(\zeta(t)) \ast F_{By,Sy}(\zeta(t))\)

\(\forall \) all \(x, y \in X, t > 0, \zeta \in Z\) and for some \(\alpha \in (1, 2)\);
(iii) the ordered pair \((A,S)\) is compatible and \(fB;Tg\) is weakly compatible or vice-versa;
(iv) one mapping of the compatible pair is continuous.

Then \(A, B, S\) and \(T\) have a unique common fixed point in \(X\).

Now we extend the above one to six self mappings and prove under weaker conditions.
**Theorem 3.2:** Let \( A, B, S, T, L \) and \( M \) be self mappings on a Menger space \((X, F, \ast)\), where \( \ast \) is the min t-norm and satisfying:

(i) \( AL(X) \subseteq T(X) \) and \( BM(X) \subseteq S(X) \);
(ii) \( F_{ALxBMx}(t) \geq F_{ALxBMx}(\zeta(t)) \ast F_{ALxBMx}(\zeta(t)) \ast F_{BMxTy}(\zeta(t)) \ast F_{ALxTV}(\alpha \zeta(t)) \ast F_{BMxSy}(\alpha \zeta(t)) \)

for all \( x, y \in X, t > 0, \zeta \in \mathbb{Z} \) and for some \( \alpha \in (1, 2) \);
(iii) the pair \( \{ AL, S \} \) and \( \{ BM, T \} \) are weakly compatible;
(iv) \( AL=LA \) and either \( SA=AS \) or \( SL=LS \);
(v) \( BM=MB \) and either \( TB=BT \) or \( TM=MT \);
(vi) the pair \( \{ AL, S \} \) and \( \{ BM, T \} \) share one of the CLR \( AL \), CLR \( BM \), CLR \( S \), CLR \( T \)-property.

Then \( A, B, S, T, L \) and \( M \) have a unique common fixed point in \( X \).

**Proof:**

**Case I:** Suppose \( \{ AL, S \} \) and \( \{ BM, T \} \) share CLR \( AL \) or CLR \( S \)-property. So, by the definition, there exist sequences \( \{ x_n \} \) and \( \{ y_n \} \) in \( X \) such that

\[ \lim_{n \to \infty} ALx_n = \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} BMy_n = \lim_{n \to \infty} Ty_n = Su, \quad \text{for some } u \in X. \]

Taking \( x = u \) and \( y = y_n \) in (ii), we get that

\[ F_{ALuBMu}(t) \geq F_{SuTy}(\zeta(t)) \ast F_{ALuSu}(\zeta(t)) \ast F_{BMuTy}(\zeta(t)) \ast F_{ALuTv}(\alpha \zeta(t)) \ast F_{BMuSu}(\alpha \zeta(t)) \]

As \( n \to \infty \), we get that

\[ F_{ALuSu}(\zeta(t)) \geq F_{ALuSu}(\alpha \zeta(t)) \geq F_{ALuSu}(\zeta(t)) \geq F_{ALuSu}(\zeta(t)) \to 1 \] as \( n \to \infty \) (since \( 0 < (\alpha - 1) < 1 \)).

Hence \( ALu=Su \).

Since \( AL(X) \subseteq T(X) \), there is a \( v \in X \) such that \( ALu = Tv \).

Taking \( x = x_n \) and \( y = v \) in (ii), we get that

\[ F_{ALuBMv}(t) \geq F_{SuTv}(\zeta(t)) \ast F_{ALuSu}(\zeta(t)) \ast F_{BMvTv}(\zeta(t)) \ast F_{ALuTv}(\alpha \zeta(t)) \ast F_{BMvSu}(\alpha \zeta(t)) \]

As \( n \to \infty \), we get that

\[ F_{BMvTv}(\zeta(t)) \geq F_{BMvTv}(\alpha \zeta(t)) \geq F_{BMvTv}(\zeta(t)) \to 1 \] as \( n \to \infty \).

Then \( BMv=Tv \). Thus \( ALu=Su=BMv=Tv=z \) (say).

Since \( AL(X) \subseteq T(X) \), there exist \( x \in X \) such that \( ALx = Tx \).

Taking \( x = x_n \) and \( y = y_n \) in (ii), we get that

\[ F_{ALxBMx}(t) \geq F_{TxTy}(\zeta(t)) \ast F_{ALxTy}(\zeta(t)) \ast F_{BMxTv}(\zeta(t)) \ast F_{ALxTv}(\alpha \zeta(t)) \ast F_{BMxSy}(\alpha \zeta(t)) \]

Similarly, by taking \( x = u \) and \( y = z \) in (ii), we get that \( BMz = z \).
Thus \( ALz = BMz = Sz = Tz = z \).

Suppose \( SA = AS \).
Since \( AL = LA \), we have \( AALz = A Az = Az \) and \( SAz = ASz = Az \).
Taking \( x = Az \) and \( y = v \) in (ii), we get that \( Az = z \).
Since \( ALz = z \), follow that \( Lz = z(\implies Az = Lz = Sz = z) \).

Suppose \( SL = LS \).
Since \( AL = LA \), we have \( ALLz = LALz = Lz \) and \( SLz = LSz = Lz \).
Taking \( x = Lz \) and \( y = v \) in (ii), we get that \( Az = z \).
Since \( ALz = z \), follow that \( Az = z(\implies Az = Lz = Sz = z) \).

Suppose \( TB = BT \).
Since \( BM = MB \), we have \( BMBz = BBMz = Bz \) and \( TBz = BTz = Bz \).
Taking \( x = Bz \) and \( y = v \) in (ii), we get that \( Bz = z \).
Since \( BMz = z \), follow that \( Bz = z(\implies Bz = Mz = Tz = z) \).

Suppose \( TM = MT \).
Since \( BM = MB \), we have \( BMMz = MBMz = Mz \) and \( TMz = MTz = Mz \).
Taking \( x = Mz \) and \( y = v \), in (ii), we get that \( Mz = z \).
Since \( BMz = z \), follow that \( Bz = z(\implies Bz = Mz = Tz = z) \).
Thus \( ALz = BMz = Sz = Tz = z \).

**Case II:** Suppose \( \{AL, S\} \) and \( \{BM, T\} \) share CLRBM or CLRr-property. So, by the definition, there exist sequences \( \{x_n\} \) and \( \{y_n\} \) in \( X \) such that
\[
\lim_{n \to \infty} ALx_n = \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} BMy_n = \lim_{n \to \infty} Ty_n = Tv, \quad \text{for some } v \in X.
\]
Proceeding on similar lines of the previous case, we first get \( BMv = Tv \). Since \( BM(X) \subseteq S(X) \), there is a \( u \in X \) such that \( BMv = Su \). Using (ii) we get that \( ALu = Su \). Thus \( ALu = Su = BMv = Tv = z \) (say). Now, using (iii), we get that \( ALz = Sz \) and \( BMz = Tz \).
Using (ii), we get that \( ALz = BMz = Sz = Tz = z \).

From this stage, the proof is same given above. Hence, we get that \( z \) is a common fixed point of \( A, B, S, T, L \) and \( M \) in \( X \).

Uniqueness follows trivially.
Hence, \( z \) is the unique common fixed point of \( A, B, S, T, L \) and \( M \) in \( X \).

Now, by taking \( L = M = \text{I}(\text{the identity mapping on } X) \), we get the following:

**Corollary 3.3:** Let \( A, B, S \) and \( T \) be self-mappings on a Menger space \((X,F,*)\), where \( * \) is the min t-norm and satisfying:

(i) \( A(X) \subseteq T(X) \) and \( B(X) \subseteq S(X) \);
(ii) \( F_{Ax,Bx}(t) \geq F_{Sx,ty}(\zeta(t))*F_{Ax,tx}(\zeta(t))*F_{Bx,ty}(\zeta(t))*F_{Ax,ty}(\alpha\zeta(t))*F_{Bx,tx}(\alpha\zeta(t)) \) for all \( x, y \in X \), \( t > 0 \), \( \zeta \in Z \) and for some \( \alpha \in (1, 2) \);
(iii) the pair \( \{A, S\} \) and \( \{B, T\} \) are weakly compatible;
(iv) the pair \( \{A, S\} \) and \( \{B, T\} \) share one of the CLRBM, CLRr, CLRr-property.

Then \( A, B, S \) and \( T \) have a unique common fixed point in \( X \).

We give an example in support of our Theorem(3.2).
Example 3.4: \((X, F, \ast)\) is a Menger space, where \(X=[0, \infty)\) with the usual metric and \(F : [0, 1] \to [0, 1]\) is defined by \(F_{x,y}(t) = \frac{t}{t + |x - y|}\) for all \(x, y \in [0, 1]\), \(t > 0\) and \(\ast\) is the min t-norm, i.e., \(a \ast b = \min\{a, b\}\) for all \(a, b \in [0, 1]\).

Define \(\zeta : [0, 1]^+ \to [0, 1]^+\) by \(\zeta(t) = 2t\).

Let \(A, B, S, T, L\) and \(M\) be the self-mappings on \(X\), defined by

\[
A(x) = \begin{cases} 
0 & \text{if } x \leq 16, \\
1 & \text{if } x > 16,
\end{cases} \\
S(x) = \begin{cases} 
0 & \text{if } x \leq 16, \\
\frac{1}{x^2} & \text{if } x > 16,
\end{cases}
\]

\(Bx = 0, Mx = \frac{x}{3}, Lx = x\) and \(Tx = x^2\), for all \(x \in X\).

Take \(x_n = \frac{1}{n}\).

Then \(\lim_{n \to \infty} ALx_n = \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} BMx_n = \lim_{n \to \infty} Ty_n = 0 = S(0)\).

So \(\{AL, S\}\) and \(\{BM, T\}\) shares CLR\(_S\)-property.

**Case 1:** When \(x \leq 16\) and \(y \in X\).

\[
\text{L.H.S} = F_{AL, BM}(t) = F_{0,0}(t) = 1.
\]

Clearly, L.H.S \(\geq\) R.H.S.

**Case 2:** When \(x > 16\) and \(y \in X\).

\[
\text{L.H.S} = F_{AL, BM}(t) = F_{1,0}(t) = \frac{t}{t + 1}.
\]

\[
\text{R.H.S} = F_{S, T}(\zeta(t)) \ast F_{AL, S}(\zeta(t)) \ast F_{BM, T}(\zeta(t)) \ast F_{AL, T}(\alpha \zeta(t)) \ast F_{BM, S}(\alpha \zeta(t))
\]

\[
= \frac{2t}{2t + \left(\frac{x}{2} - y\right)^2} \ast \frac{2t}{2t + \left(\frac{x}{2} - 1 \right)^2} \ast \frac{2t}{2t + y^2} \ast \frac{2t}{2t + 1 - y^2} \ast \frac{2t}{2t + \left(\frac{x}{2}\right)^2}
\]

\[
\leq \frac{2t}{2t + \left(\frac{x}{2} - 1 \right)^2} \leq \frac{t}{\left(\frac{x}{2} - 1 \right)^2} \leq \frac{t}{t + 1} = \text{L.H.S}.
\]

The other conditions of the theorem are trivially satisfied. Clearly, ‘0’ is the unique common fixed point of \(A, B, S, T, L\) and \(M\) in \(X\).

**Remark 3.5:** If we assume JCLR-property instead of CLR we can omit condition (i) in Theorem(3.2).

We now prove the following:

**Theorem 3.6:** Let \(A, B, S, T, L\) and \(M\) be self-mappings on a Menger space \((X, F, \ast)\), where \(\ast\) is the min t-norm and satisfying:

\[(i)\quad \text{the pair \{AL, S\} and \{BM, T\} shares JCLR}\_S\text{-property;}
\]
(ii) \( F_{AL,BM} (t) \geq F_{S,T} (\zeta(t)) * F_{AL,S} (\zeta(t)) * F_{BM,T} (\zeta(t)) * F_{AL,T} (\alpha \zeta(t)) * F_{BM,S} (\alpha \zeta(t)) \)
for all \( x, y \in X, t > 0, \zeta \in Z \) and for some \( \alpha \in (1, 2) \);
(iii) the pair \( \{ AL, S \} \) and \( \{ BM, T \} \) are weakly compatible;
(iv) \( AL=LA \) and either \( SA=AS \) or \( SL=LS \);
(v) \( BM=MB \) and either \( TB=BT \) or \( TM=MT \).
Then \( A, B, S, T, L \) and \( M \) have a unique common fixed point in \( X \).

**Proof:**

Suppose \( \{ AL, S \} \) and \( \{ BM, T \} \) shares JCLR\(_{ST}\)-property. Then there exist sequences \( \{ x_n \} \) and \( \{ y_n \} \) in \( X \) and \( u \in X \) such that \( Su = Tu \) and

\[
\lim_{n \to \infty} ALx_n = \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} BMy_n = \lim_{n \to \infty} Ty_n = Su (= Tu).
\]

Taking \( x = u \) and \( y = y_n \) in (ii), we get that \( ALu = Su \) (\( ALu = Su = Tu \)).

Taking \( x = x_n \) and \( y = u \) in (ii), we get that \( Tu = BMu \).

Thus \( ALu = Su = BMv = Tv = z \) (say).

Since \( \{ AL, S \} \) and \( \{ BM, T \} \) are weakly compatible, \( ALSu = SALu \) and \( BMTu = TBMu \). i.e, \( ALz = Sz \) and \( BMz = Tz \).

Taking \( x = z \) and \( y = u \) in (ii), we get that \( ALz = Sz = z \).

Taking \( x = u \) and \( y = z \) in (ii), we get that \( BMz = Tz = z \).

Hence, \( ALz = Sz = BMz = Tz = z \).

From this stage, the proof runs as in Theorem (3.2).

We now give an example in support of our Theorem(3.6).

**Example 3.7:** \((X, F, \ast)\) is a Menger space, where \( X = [0, \infty) \) with the usual metric and \( F : \mathbb{I} \to [0, 1] \) is defined by \( F_{x,y} (t) = \frac{t}{t + |x - y|} \) for all \( x, y \in \mathbb{I}, t > 0 \) and \( \ast \) is the min \( t \)-norm, i.e, \( a \ast b = \min\{a, b\} \) for all \( a, b \in [0, 1] \).

Define \( \zeta : \mathbb{I} \to \mathbb{I} \) by \( \zeta(t) = 2t \).

Let \( A, B, S, T, L \) and \( M \) be the self-mappings on \( X \), defined by

\[
A(x) = \begin{cases} 
0 & \text{if } x \leq 25, \\
1 & \text{if } x > 25,
\end{cases}
S(x) = \begin{cases} 
0 & \text{if } x \leq 25 \\
\frac{1}{x^2} & \text{if } x > 25,
\end{cases}
\]

\( Bx = 0, Mx = \frac{x}{T}, Lx = x \) and \( Tx = x^2 \), for all \( x \in X \).

Take \( x_n = y_n = \frac{1}{n} \). Then \( \lim ALx_n = \lim Sx_n = \lim BMy_n = \lim Ty_n = 0 = S(0) = T(0) \).

So \( \{ AL, S \} \) and \( \{ BM, T \} \) shares JCLR\(_{ST}\)-property.

**Case 1:** When \( x \leq 25 \) and \( y \in X \).

L.H.S: \( F_{AL,BM} (t) = F_{0,0} (t) = 1 \).

Clearly, L.H.S \( \geq \) R.H.S.

**Case 2:** When \( x > 25 \) and \( y \in X \).
L.H.S = \( F_{\text{ALx}, B\text{My}} (t) = F_{1,0} (t) = \frac{t}{t + 1} \).

R.H.S = \( F_{\text{Sx}, T\text{y}} (\zeta(t)) \ast F_{\text{ALx}, S\text{y}} (\zeta(t)) \ast F_{\text{BMy}, T\text{y}} (\zeta(t)) \ast F_{\text{ALx}, T\text{y}} (\alpha \zeta(t)) \ast F_{\text{BMy}, S\text{x}} (\alpha \zeta(t)) \)

\[ = F_{x^{1/2}, y} (2t) \ast F_{1, x^{1/2}} (2t) \ast F_{0, y} (2t) \ast F_{1, y^2} (2t) \ast F_{0, x^{1/2}} (2t) \]

\[ = \min \left\{ \frac{2t}{2t + \left( x^{1/2} - y^2 \right)}, \frac{2t}{2t + (x^{1/2} - 1)}, \frac{2t}{2t + y^2}, \frac{2\alpha t}{2\alpha t + \left| 1 - y^2 \right|}, \frac{2\alpha t}{2\alpha t + x^{1/2}} \right\} \]

\[ \leq \frac{2t}{2t + (x^{1/2} - 1)} \]

\[ \leq \frac{t}{t + \left( x^{1/2} - 1 \right)} \]

\[ \leq \frac{t}{t + 1} = \text{L.H.S.} \]

The other conditions of the theorem are trivially satisfied. Clearly, ‘0’ is the unique common fixed point of A, B, S, T, L and M in X.

We have the following theorem where the pairs share common property (E.A). The proof runs on similar lines of Theorem (3.2).

**Theorem 3.8:** Let \( A, B, S, T, L \), and \( M \) be self-mappings on a Menger space \((X,F, *)\), where \(*\) is the \( \text{min} \) \( t \)-norm and satisfying:

(i) \( A(X) \subseteq T(X) \) and \( B(X) \subseteq S(X) \);

(ii) \( F_{\text{ALx}, B\text{My}} (t) \geq F_{\text{Sx}, T\text{y}} (\zeta(t)) \ast F_{\text{ALx}, S\text{y}} (\zeta(t)) \ast F_{\text{BMy}, T\text{y}} (\zeta(t)) \ast F_{\text{ALx}, T\text{y}} (\alpha \zeta(t)) \ast F_{\text{BMy}, S\text{x}} (\alpha \zeta(t)) \)

for all \( x, y \in X, t > 0, \zeta \in Z \) and for some \( \alpha \in (1, 2) \);

(iii) the pair \{AL, S\} and \{BM, T\} are weakly compatible;

(iv) \( AL=LA \) and either \( SA=AS \) or \( SL=LS \);

(v) \( BM=MB \) and either \( TB=BT \) or \( TM=MT \);

(vi) the pair \{AL, S\} and \{BM, T\} shares common property (E.A);

(vii) one of \( AL(X), BM(X), S(X), T(X) \) is a complete subspace of \( X \).

Then \( A, B, S, T, L \), and \( M \) have a unique common fixed point in \( X \).

Example 3.9: \((X, F, *)\) is a Menger space, where \( X=[0, \infty) \) with the usual metric and \( F : [0, \infty) \rightarrow [0,1] \) is defined by \( F_{x,y} (t) = \frac{t}{t + |x - y|} \) for all \( x, y \in [0, \infty) \), \( t > 0 \) and \(*\) is the \( \text{min} \) \( t \)-norm, i.e, \( a \ast b = \min\{a,b\} \) for all \( a, b \in [0,1] \).

Define \( \zeta : [0, \infty) \rightarrow [0, \infty) \) by \( \zeta(t) = 2t \).

Let \( A, B, S, T, L \) and \( M \) be the self-mappings on \( X \), defined by

\[ A(x) = \begin{cases} 0 & \text{if } x \leq 9, \\ 1 & \text{if } x > 9, \end{cases} \quad S(x) = \begin{cases} 0 & \text{if } x \leq 9, \\ \frac{1}{x^2} & \text{if } x > 9, \end{cases} \]

\[ Bx = 0, Mx = \frac{x}{5}, Lx = x \text{ and } Tx = x^2, \text{ for all } x \in X. \]

Take \( x_n = y_n = \frac{1}{n} \).
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References


