Resolvability in generalized double Topological Space

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Abstract
In this paper the relationship between resolvability in generalized double topological spaces and pairwise resolvability in bigeneralized topological spaces has been studied.

1. Introduction:
In 1968 C.L. Chang [2] introduced the concept of intuitionistic fuzzy topological space. The concept of intuitionistic fuzzy set was introduced in 1986 by Atanassov [1] as a possible generalization of ordinary fuzzy sets. In 1997 coker [3] introduced the concept of intuitionistic fuzzy topological space. In 1989, Kandil introduced the concept of fuzzy bitopological space. The concepts of resolvability and irresolvability in a topological space were introduced and studied by E. Hewit in 1943.

2. Preliminaries
Now we introduce some basic definitions. Throughout the remainder of this paper we use the simpler notation A = (A₁, A₂) for a double set.

Definition 2.1 [4]
A double set A is an object having the form A = <x, A₁, A₂> where A₁ and A₂ are subsets of X satisfying A₁ ∩ A₂ = ∅. The set A₁ is called the set of members of A while A₂ is the set of non-members of A.

Definition 2.2 [4]
Let the double sets A and B on X be of the form A = (A₁, A₂), B = (B₁, B₂) respectively. Furthermore, let {Aₖ: j ∈ J} be an arbitrary family of double sets in X, where Aₖ = (Aₖ⁽¹⁾, Aₖ⁽²⁾). Then
(a) A ⊆ B if and only if A₁ ⊆ B₁ and A₂ ⊆ B₂
(b) A = B if and only if A ⊆ B and B ⊆ A
(c) A̅ = (A₂, A₁) denotes the complement of A
(d) ∩ Aₖ = (∩Aₖ⁽¹⁾, ∪Aₖ⁽²⁾)

(e) ∪ Aₖ = (∪Aₖ⁽¹⁾, ∩Aₖ⁽²⁾)
(f) ∅ = (∅, X) and X = (X, ∅)

Definition 2.3 [5]
A generalized double topology on a set X is a family τ of double sets in X satisfying the following axioms

T₁ : ∅ ∈ τ
T₂ : ∪ Gⱼ ∈ τ for any arbitrary family {Gⱼ : j ∈ J} ⊆ τ

In this case the pair (X, τ) is called a generalized double topological space and double set in τ is known as a double open set. The complement A̅ of an double open set A in generalized double topological space is called a double closed set in X.

Definition 2.4 [5]
Let (X, τ) be generalized double topological space and A = (A₁, A₂) be double set in X. Then the interior and closure of A are defined by

int (A) = ∪ { G: G is a double open sets in X and G ⊆ A}
cl (A) = ∩ { H: H is a double closed sets in X and A ⊆ H}

respectively.

3. Comparison of resolvability in generalized double topology with bigeneralized topology:

Definition: 3.1
A generalized topological space (X, T) is called resolvable if there exist a dense set A in (X, T) such that X − A is also a dense in (X, T). Otherwise (X, T) is called a irresolvable space.

Definition: 3.2
A bigeneralized topological space (X, T₁, T₂) is called a pair wise resolvable space if there exists a T₁ dense set A such that X − A is a T₂ dense set or a T₂ dense set B such that X − B is a T₁ dense set. Otherwise (X, T₁, T₂) is called a pair wise irresolvable space.

Definition: 3.3
A double set A in a generalized double topological space (X, T) is called dense if there exist no double closed set B such that A ⊆ B ⊆ X.
**Definition: 3.4**

A generalized double topological space \((X, T)\) is said to be resolvable if there exists a dense double set \(A\) in \((X, T)\) whose complement is also dense in \((X, T)\).

**Definition: 3.5**

A generalized double topological space \((X, T)\) is said to be irresolvable if it is not resolvable.

**Theorem: 3.6**

Let \((X, T)\) be a generalized double topological space. Let \(T = \{ (A_i, B_i) \}\). Let \(T_1 = \{ \text{sets formed by the first co-ordinates of elements of } T \} \), i.e., \(T_1 = \{ A_i \}\). Then \(T_1\) is a generalized topological space.

**Proof:**

Since \((\phi, X)\) and \((X, \phi)\) \(\in T\), \(\phi, X \in T_1\).

Also \(T_1\) is closed under arbitrary union.

Hence for any collection of open sets \(\{ (A_i, B_i) \}\) in \(T\), \(\cup (A_i, B_i) = (\cup A_i \cap B_i) \in T\). Therefore, \(\cup A_i \in T_1\), i.e., any collection of \(\{ A_i \}\) in \(T_1\) is closed under arbitrary union.

Hence \(T_1\) forms a generalized topological space.

**Theorem: 3.7**

Let \((X, T)\) be a generalized double topological space such that for any open set \((A_i, B_i)\), \(A_i \neq \phi\). Let \(T_1\) be the generalized topological space formed by the first co-ordinates. If \((X, T_1)\) is resolvable then \((X, T)\) is resolvable.

**Proof:**

Let \(T = \{ (A_i, B_i) \}\) be the generalized double topology. Let \((X, T_1)\) be resolvable where \(T_1\) is the first co-ordinate generalized topology. Then there exist \(A\) such that \(A\) and \(X - A\) are dense in \((X, T_1)\).

Since \(A\) is dense in \((X, T_1)\), \(\forall i \in X - A_i \neq X, A \subset X - A_i\), i.e., \(\forall i\) and \(A_i \neq \phi\).

That implies \(\forall i, B_i \neq X, A \supset B_i\).

Therefore, \((A, B) \subset (B_i, A_i)\) \(\forall i\) and \(B_i \neq X\) and for any \(B\) such that \(A \cap B = \phi\).

Hence \((A, B)\) is dense in \((X, T)\).

Now, since \(X - A\) is dense in \((X, T_1)\), \(\forall i\) and \(X - A_i \neq X, X - A \subset X - A_i\), i.e., \(\forall i\) and \(A_i \neq \phi, X - A \supset X - A_i\), i.e., \(A_i \subset A_i\). Hence \(\forall i\) and \(A_i \neq \phi, A \supset A_i\). Hence \((B, A) \subset (B_i, A_i)\) \(\forall i\) and \(A_i \neq \phi\). Hence \((B, A)\) is dense in \((X, T)\). Therefore, \((X, T)\) is resolvable.

**Result: 3.8**

The converse of the above theorem is not true.

Let \(X = \{a, b, c\}\). Let \(T = \{ \phi, ([c], [a]), ([a, b], [c]) \}\). Clearly \(T\) is a generalized double topological space. Here \((X, T)\) is resolvable space.

But \((X, T_1)\) is not resolvable.

**Result: 3.9**

Theorem 3.7 is not true when there is an open set \((A_1, A_2) \neq (\phi, X)\) such that \(A_1 = \phi\) consider the following example.

Let \(X = \{a, b\}\) and \(T = \{ \phi, (\phi, \{a\}), X \}\). Here \(T_1 = \{ \phi \}, X\). Hence \((X, T)\) is obviously resolvable. But \((X, T)\) is a irresolvable space.

**Theorem: 3.10**

Let \((X, T)\) be a generalized double topological space. Let \(T = \{ (A_i, B_i) \}\). Let \(T_2 = \{ \text{sets formed by complement of second co-ordinate of elements of } T \}\), i.e., \(T_2 = \{ X - B_i \}\). Then \(T_2\) forms a generalized topology.

**Proof:**

Since \((\phi, X)\) and \((X, \phi)\) \(\in T\), \(X, \phi \in T_2\).

For any arbitrary collection of sets \(\{ X - B_i \}\) in \(T_2\), \(\cup \{ X - B_i \} = X - \cap B_i\). Since \(T\) is closed under arbitrary union, \(\cup (A_i, B_i) = (\cup A_i \cap B_i) \in T\).

Hence \(\cup B_i \in T_1\). Therefore \(T_2\) is closed under arbitrary union. Hence \(T_2\) is a generalized topology.

**Theorem: 3.11**

Let \((X, T)\) be a generalized double topological space. Let \(T_2\) be the generalized topology formed by the complement of second co-ordinates of \(T\). If \((X, T_2)\) is resolvable then \((X, T)\) is resolvable.

**Proof:**

Let \(T = \{ (A_i, B_i) \}\) be the generalized double topology. Let \(T_2\) be the second co-ordinate generalized topology and \((X, T_2)\) be resolvable. Then there exist a set \(A\) such that \(A\) and \(X - A\) are dense in \((X, T_2)\).

Since \(A\) is dense in \((X, T_2)\), \(\forall i\) and \(B_i \neq X, A \subset B_i\). Therefore \(\forall i\) and \(B_i \neq X, (A, X - A) \subset (B_i, A_i)\).

Therefore \((A, X - A)\) is dense in \((X, T)\).

Also since \(X - A\) is dense in \((X, T_2)\), \(X - A \subset B_i\). Therefore \(\forall i\) and \(B_i \neq X, (A, X - A) \subset (B_i, A_i)\).

Hence \((X, T)\) is resolvable.

**Result: 3.12**

\((X, T)\) is resolvable does not imply that \((X, T_2)\) is resolvable, where \(T_2\) is the generalized
topology formed by the complement of second components of T. Consider the example.

Let \( X = \{ a, b, c \} \)

Let \( T = \{ \emptyset, X, \{ c \}, \{ b \}, \{ a \} \} \)

Here \((X, T)\) is resolvable. But \((X, T_2)\) is irresolvable

**Definition: 3.13**

The Let \((X, T)\) be generalized double topological space. Then \((X, T_1, T_2)\) is called the induced bigeneralized topological space where \(T_1\) and \(T_2\) are generalized topologies formed by the first and the complement of second co-ordinates of \(T\) respectively.

**Theorem: 3.14**

Let \((X, T)\) be an generalized double topological space where \(T = \{\{A_i, B_i\}\}\) such that for any non-empty open set \((A_i, B_i)\), \(A_i \neq \emptyset\). If the induced bigeneralized topological space \((X, T_1, T_2)\) is pair wise resolvable, then \((X, T)\) is resolvable.

**Proof:**

Let the induced bigeneralized topological space \((X, T_1, T_2)\) be pair wise resolvable. Then there exist a dense set \(A\) in \(T_1\) such that \(X - A\) is dense in \(T_2\). ie, \(\forall i\) and \(X - A_i \neq X\). And \(\forall i\) and \(B_i \neq X\). \(X - A \supsetneq B_i\). Hence \(\forall i\) and \(A_i \neq \emptyset\) and \(B_i \neq X\). \((X - A, A) \subset (B_i, A_i)\). Therefore, \((X - A, A)\) is dense in \((X, T)\)

Also \(\forall i\) and \(X - A_i \neq X\). \(A_i \supsetneq X - A_i\)

Moreover \(A_i \cap B_i = \emptyset\). Therefore \(B_i \subseteq X - A_i\). Hence \(\forall i\) and \(X - A_i \neq X\). \(A_i \supsetneq B_i\). So \((A_i, X - A) \subset (B_i, A_i)\). \(\forall i\) Hence \((A_i, X - A)\) is dense in \((X, T)\).

Therefore, \((X, T)\) is resolvable space.

**Result: 3.15**

Converse of the above theorem is not true. For the example, let \(X = \{ a, b, c \} \)

Let \(T = \{ \emptyset, X, \{ c \}, \{ b \}, \{ a \}, \{ \phi \} \} \).

Here \((X, T)\) is resolvable, but \((X, T_1, T_2)\) is not pair wise resolvable.

Now to find the condition when the converse of the above theorem is true.

**Theorem: 3.16**

Let \((X, T)\) be a resolvable space. Let \((\rho, \lambda)\) be a double dense set whose complement is also dense. Let \(T = \{A_i, B_i\}\) be the collection of open sets. If \(A_i \subseteq \rho \subseteq B_i\) \(\forall i\) or \(A_i \subseteq \lambda \subseteq B_i\) \(\forall i\) then \((X, T_1, T_2)\) is pair wise resolvable.

**Proof:**

Let \(A_i \subseteq \rho \subseteq B_i\) \(\forall i\). Since \((\rho, \lambda)\) dense in \((X, T), (\rho, \lambda) \subset (B_i, A_i) \forall i\)

Therefore \(\forall i\) \(\rho \subset B_i\) or \(\lambda \supset A_i\)

ie, \(\forall i\), \(\rho \supset B_i\) or \(\lambda \subset A_i\). But \(\rho \subseteq B_i\) \(\forall i\).

Hence \(\forall i\) \(\lambda \subset A_i\) \(\forall i\). \(\lambda \subset X - A_i\) \(\forall i\).

Hence \(X - \lambda \subset X - A_i \forall i\).

That implies \(X - \lambda \subset X - A_i \forall i\).

Hence \(X - \lambda\) is dense in \(T_1\).

Now since \((\lambda, \rho)\) is dense in \((X, T), (\lambda, \rho) \subset (B_i, A_i) \forall i\)

Therefore \(\forall i\), \(\lambda \subset B_i\) or \(\rho \supset A_i\)

ie, \(\forall i\), \(\lambda \subset B_i\) or \(\rho \subset A_i\). But \(A_i \subseteq \rho \forall i\). Hence \(\lambda \supset B_i \forall i\).

So \(\lambda\) is dense in \(T_2\). Hence \((X, T_1, T_2)\) is pair wise resolvable.

The proof is similar if \(A_i \subseteq \lambda \subseteq B_i\) \(\forall i\)

**References:**


