On general Eulerian integral of certain products of A-functions and a multivariable Aleph-function

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ABSTRACT
The object of this paper is to establish a general Eulerian integral involving the product of two multivariable A-functions defined by Gautam et al [1], the multivariable Aleph-function and a generalized hypergeometric function which provide unification and extension of numerous results. We will study the particular case concerning the multivariable H-function.

Keywords: Eulerian integral, multivariable A-function, Lauricella function of several variables, multivariable H-function, generalized hypergeometric function.

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1. Introduction
In this paper, we evaluate a new Eulerian integral of most general characters associated with the products of two multivariable A-functions defined by Gautam et al [1], the Aleph-function of several variables and a generalized hypergeometric function with general arguments. The Aleph-function of several variables is an extension of multivariable I-function defined by Sharma et al [3].

We define:

\[ N(z_1^m, \cdots, z_v^m) = \prod_{i=1}^{N} \frac{1}{\Gamma(a_{ji} + 1, n_i + 1, p_i)} \left( \frac{z_i^{\alpha_{ji}(v)} - 1}{z_i^{\beta_{ji}(v)} - 1} \right) \]

and

\[ (a_j, \alpha_j(1), \cdots, \alpha_j(v))_{1,n} : \]

\[ \tau_i(\alpha_{ji}(v))_{n+1,p_i} : \]

\[ (c_j(1), \gamma_j(1))_{1,n_i} : \]

\[ \tau_i(c_j(1), \gamma_j(1))_{m+1,q_i} : \]

\[ (d_j(1), \delta_j(1))_{1,m_i} : \]

\[ \tau_i(d_j(1), \delta_j(1))_{m+1,q_i} : \]

\[ = \frac{1}{(2\pi i)^v} \int_{L_1} \cdots \int_{L_v} \psi_1(s_1, \cdots, s_v) \prod_{k=1}^{v} \xi_k(s_k) z_1^{m/s_1} ds_1 \cdots ds_v \]  

(1.1)

with \( \omega = 1 \)

\[ \psi_1(s_1, \cdots, s_v) = \prod_{j=1}^{N} \Gamma(1 - a_{ji} + \sum_{k=1}^{v} \alpha_j(k) s_k) \]

\[ \sum_{i=1}^{R} \left[ \tau_i(1) \Gamma(a_{ji} - \sum_{k=1}^{v} \alpha_j(k) s_k) \prod_{j=1}^{N} \Gamma(1 - b_{ji} + \sum_{k=1}^{v} \beta_j(k) s_k) \right] \]

(1.2)

and

\[ \xi_k(s_k) = \frac{\prod_{j=1}^{M} \Gamma(d_j(k) - \delta_j(k) s_k) \prod_{j=M+1}^{N} \Gamma(1 - d_j(k) + \gamma_j(k) s_k) \prod_{j=M+1}^{N} \Gamma(c_j(k) - \gamma_j(k) s_k)}{\prod_{i=1}^{R} \tau_i(1) \prod_{j=M+1}^{N} \Gamma(1 - d_j(k) + \delta_j(k) s_k) \prod_{j=M+1}^{N} \Gamma(c_j(k) - \gamma_j(k) s_k)} \]

(1.3)

Suppose, as usual, that the parameters

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\( \alpha_j, j = 1, \cdots, P; b_j, j = 1, \cdots, Q; \)
\( c_j^{(k)}, j = 1, \cdots, N_k; c_j^{(k)} / c_j^{(k)}, j = N_k + 1, \cdots, P^{(k)}; \)
\( d_j^{(k)}, j = 1, \cdots, M_k; d_j^{(k)} / d_j^{(k)}, j = M_k + 1, \cdots, Q^{(k)}; \)

with \( k = 1, \cdots, r, i = 1, \cdots, R, ^{(k)} = 1, \cdots, R^{(k)} \)

are complex numbers, and the \( \alpha' s, \beta' s, \gamma' s \) and \( \delta' s \) are assumed to be positive real numbers for standardization purpose such that

\[
U^{(k)}_i = \sum_{j=1}^{N} \alpha_j^{(k)} + \tau_i \sum_{j=N+1}^{P} \alpha_j^{(k)} + \sum_{j=1}^{N_k} \gamma_j^{(k)} + \tau'_i \sum_{j=N_k+1}^{P^{(k)}} \gamma_j^{(k)} - \tau_i \sum_{j=1}^{Q_i} \beta_j^{(k)} - \sum_{j=1}^{M_k} \delta_j^{(k)} - \tau'_i \sum_{j=M_k+1}^{Q^{(k)}} \delta_j^{(k)} \leq 0
\]  

(1.4)

The real numbers \( \tau_i \) are positives for \( i = 1 \) to \( R \), \( \tau'_i \) are positives for \( i^{(k)} = 1 \) to \( R^{(k)} \)

The contour \( L_k \) is in the \( s_k \)-plane and run from \( \sigma - i \infty \) to \( \sigma + i \infty \) where \( \sigma \) is a real number with loop, if necessary, ensure that the poles of \( \Gamma(d_j^{(k)} - \delta_j^{(k)} s_k) \) with \( j = 1 \) to \( M_k \) are separated from those of \( \Gamma(1 - \alpha_j + \sum_{i=1}^{r} \alpha_j^{(k)} s_k) \) with \( j = 1 \) to \( n \) and \( \Gamma(1 - \alpha_j + \gamma_j^{(k)} s_k) \) with \( j = 1 \) to \( N_k \) to the left of the contour \( L_k \). The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as:

\[
\arg z^{(k)}_k < \frac{1}{2} A^{(k)}_i \pi, \text{ where }
\]

\[
A^{(k)}_i = \sum_{j=1}^{N} \alpha_j^{(k)} - \tau'_i \sum_{j=N+1}^{P} \alpha_j^{(k)} - \tau_i \sum_{j=1}^{Q_i} \beta_j^{(k)} + \sum_{j=1}^{N_k} \gamma_j^{(k)} - \tau'_i \sum_{j=N_k+1}^{P^{(k)}} \gamma_j^{(k)} + \sum_{j=1}^{M_k} \delta_j^{(k)} - \tau'_i \sum_{j=M_k+1}^{Q^{(k)}} \delta_j^{(k)} > 0, \text{ with } k = 1, \cdots, r, i = 1, \cdots, R, ^{(k)} = 1, \cdots, R^{(k)} \]  

(1.5)

The complex numbers \( z_k \) are not zero. Throughout this document, we assume the existence and absolute convergence conditions of the multivariable Aleph-function.

We may establish the asymptotic expansion in the following convenient form:

\[
\mathcal{N}(z^{(k)}_1, \cdots, z^{(k)}_n) = 0(\max(|z_1^{(k)}|, \cdots, |z_n^{(k)}|), \max(|z_1^{(k)}|, \cdots, |z_n^{(k)}|) \to 0
\]

\[
\mathcal{N}(z^{(k)}_1, \cdots, z^{(k)}_n) = 0(\max(|z_1^{(k)}|, \cdots, |z_n^{(k)}|), \max(|z_1^{(k)}|, \cdots, |z_n^{(k)}|) \to \infty
\]

where \( k = 1, \cdots , r : \alpha_k = \min(Re(d_j^{(k)} / \delta_j^{(k)})), j = 1, \cdots, m_k \) and

\[
\beta_k = \max(Re((c_j^{(k)} - 1) / \gamma_j^{(k)})), j = 1, \cdots, n_k
\]

Serie representation of Aleph-function of \( u \)-variables is given by
Where \( \psi(\cdot, \cdot, \cdot, \cdot, \cdot, \cdot, \cdot) \), \( \theta_i(\cdot), i = 1, \cdot, r \) are given respectively in (1.2), (1.3) and

\[
\eta_{G_1, g_1} = \frac{d_{g_1}^{(1)} + G_1}{d_{g_1}^{(1)}}, \ldots, \eta_{G_v, g_v} = \frac{d_{g_v}^{(v)} + G_v}{d_{g_v}^{(v)}}
\]

which is valid under the conditions \( \delta^{(i)}_{g_i}[d_{g_i}^{(i)} + p_i] \neq \delta^{(j)}_{g_j}[d_{g_j}^{(j)} + G_j] \)

for \( j \neq m_i, m_i = 1, \cdot, \cdot, r, g_i, n_i = 0, 1, 2, \cdot, \cdot, y_i \neq 0, i = 1, \cdot, \cdot, v \)

The A-function is defined and represented in the following manner.

\[
A(z_1', \cdot, \cdot, z_s') = A_{m', n', m_1', n_1'; \cdot, \cdot, m_s', n_s'}^{p', q', p_1'; \cdot, \cdot, q_s'} \left[ \begin{array}{c} z_1' \\ \cdot \\ \cdot \\ z_s' \end{array} \right] \left( \begin{array}{c} a^{(1), 1}_j, \cdot, \cdot, A^{(s), 1}_j \\ \cdot \\ \cdot \\ b^{(1), 1}_j, \cdot, \cdot, B^{(s), 1}_j \end{array} \right)_{1, p'}^{1, q'}
\]

\[
\begin{array}{l}
(c_1^{(1)}, C_1^{(1)})_{1, p_1'}; \cdot, \cdot, (c_s^{(s)}, C_s^{(s)})_{1, p_s'} \\
(d_1^{(1)}, D_1^{(1)})_{1, q_1'}; \cdot, \cdot, (d_s^{(s)}, D_s^{(s)})_{1, q_s'}
\end{array}
\]

\[
= \frac{1}{(2\pi)^s} \int_{L_1} \cdots \int_{L_s} \phi(t_1, \cdot, \cdot, t_s) \prod_{i=1}^{s} \zeta_i(t_i) z_i^{m_i} dt_1 \cdots dt_s
\]

where \( \phi(t_1, \cdot, \cdot, t_s), \zeta_i(t_i), i = 1, \cdot, s \) are given by :

\[
\phi(t_1, \cdot, \cdot, t_s) = \frac{\prod_{j=1}^{n'} \Gamma(b_j - \sum_{i=1}^{s} B^{(i), (i)}_j t_i) \prod_{j=1}^{m'} \Gamma(1 - a_j' + \sum_{i=1}^{s} A^{(i), (i)}_j t_j)}{\prod_{j=n'+1}^{m'} \Gamma(a_j' - \sum_{i=1}^{s} A^{(i), (i)}_j t_j) \prod_{j=m'+1}^{m''} \Gamma(1 - b_j' + \sum_{i=1}^{s} B^{(i), (i)}_j t_j)}
\]

\[
\zeta_i(t_i) = \frac{\prod_{j=1}^{n_i} \Gamma(1 - c_j^{(i)} + C_j^{(i), (i)} t_i) \prod_{j=1}^{m_i} \Gamma(d_j^{(i)} - D_j^{(i), (i)} t_i)}{\prod_{j=n_i'+1}^{m_i'} \Gamma(c_j^{(i)} - C_j^{(i), (i)} t_i) \prod_{j=m_i'+1}^{m_i''} \Gamma(1 - d_j^{(i)} + D_j^{(i), (i)} t_i)}
\]

Here \( m', n', p', m_1', n_1', p_1', c_i' \in \mathbb{N}^*; i = 1, \cdot, \cdot, s; a_j', b_j', c_j^{(i)}', d_j^{(i)}', A_j^{(i)}, B_j^{(i)}, C_j^{(i)}, D_j^{(i)} \in \mathbb{C} \)

The multiple integral defining the A-function of \( r \) variables converges absolutely if :

\[
|\arg(\Omega_k)z_k^k| < \frac{1}{2} \eta_k \pi, \xi^* = 0, \eta_k > 0
\]
Consider the second multivariable A-function.

\begin{equation}
\Omega_i = \prod_{j=1}^{p'} \left\{ A_j^{(i)} \right\} \prod_{j=1}^{q'} \left\{ B_j^{(i)} \right\} \prod_{j=1}^{q'_i} \left\{ D_j^{(i)} \right\} \prod_{j=1}^{p'_i} \left\{ C_j^{(i)} \right\}^{-i}; \quad i = 1, \cdots, s
\end{equation}

(1.13)

\begin{equation}
\xi_i = \Im \left( \sum_{j=1}^{q'} \left\{ A_j^{(i)} \right\} - \sum_{j=1}^{p'} \left\{ B_j^{(i)} \right\} \right) = \Re \left( \sum_{j=1}^{p'} \left\{ A_j^{(i)} \right\} \right) - \Re \left( \sum_{j=1}^{q'} \left\{ B_j^{(i)} \right\} \right); \quad i = 1, \cdots, s
\end{equation}

(1.14)

\begin{equation}
\eta_i = \Re \left( \sum_{j=1}^{q'_i} \left\{ A_j^{(i)} \right\} \right) + \Re \left( \sum_{j=1}^{p'_i} \left\{ B_j^{(i)} \right\} \right) - \Re \left( \sum_{j=1}^{q'_i} \left\{ D_j^{(i)} \right\} \right) - \Re \left( \sum_{j=1}^{p'_i} \left\{ C_j^{(i)} \right\} \right); \quad i = 1, \cdots, s
\end{equation}

(1.15)

Consider the second multivariable A-function.

\[ A(z''_1, \cdots, z''_u) = A_{p''}^{m''} \cdot m''(1), \cdots, m''(u) ; \cdot p'' \cdot q'' \cdot C_{j}^{(i)}(1), p_{j}^{(i)} ; \cdot \left( \begin{array}{c}
Z''_1 \\
\vdots \\
Z''_u
\end{array} \right)
\]

(1.16)

\begin{equation}
\frac{1}{(2\pi \omega)^u} \int_{L''_1} \cdots \int_{L''_u} \phi'(x_1, \cdots, x_u) \prod_{i=1}^{u} \theta_i'(x_i) z''_i \cdot \mathrm{d}x_1 \cdots \mathrm{d}x_u
\end{equation}

(1.17)

where \( \phi'(x_1, \cdots, x_u), \theta_i'(x_i), i = 1, \cdots, u \) are given by:

\begin{equation}
\phi'(x_1, \cdots, x_u) = \prod_{j=1}^{m''} \Gamma(b_j'' - \sum_{i=1}^{u} B_j^{(i)}(x_1) \prod_{j=1}^{m''} \Gamma(1 - a_j'' + \sum_{i=1}^{u} A_j^{(i)}(x_1))
\end{equation}

(1.18)

and

\begin{equation}
\theta_i'(x_i) = \prod_{j=1}^{m''} \Gamma(1 - c_j''(i) - C_j''(i) x_i) \prod_{j=1}^{m''} \Gamma(d_j''(i) - D_j''(i) x_i)
\end{equation}

(1.19)

Here \( m'', n'', p'', q'', c_i'' \in \mathbb{N}^*; i = 1, \cdots, u \); \( a_j'', b_j'', c_j''(i), d_j''(i), A_j^{(i)}, B_j^{(i)}, C_j^{(i)}, D_j^{(i)} \in \mathbb{C} \)

The multiple integral defining the A-function of \( r \) variables converges absolutely if:
2. Integral representation of generalized Lauricella function of several variables

The following generalized hypergeometric function in terms of multiple contour integrals is also required [3, page 39 eq. 30]

\[
\frac{\prod_{j=1}^{p} \Gamma(A_j)}{\prod_{j=1}^{q} \Gamma(B_j)} \, {}_pF_Q \left[ (A_P); (B_Q); -(x_1 + \cdots + x_r) \right] \nonumber
\]

\[
= \frac{1}{(2\pi \alpha)^r} \int_{L_1} \cdots \int_{L_r} \frac{\prod_{j=1}^{p} \Gamma(A_j + s_1 + \cdots + s_r)}{\prod_{j=1}^{q} \Gamma(B_j + s_1 + \cdots + s_r)} \Gamma(-s_1) \cdots \Gamma(-s_r) x_1^{s_1} \cdots x_r^{s_r} \, ds_1 \cdots ds_r \quad (2.1)
\]

where the contours are of Barnes type with indentations, if necessary, to ensure that the poles of \(\Gamma(A_j + s_1 + \cdots + s_r)\) are separated from those of \(\Gamma(-s_j), j = 1, \cdots, r\). The above result (1.23) can be easily established by an appeal to the calculus of residues by calculating the residues at the poles of \(\Gamma(-s_j), j = 1, \cdots, r\).

The Lauricella function \(F_D^{(k)}\) is defined as

\[
F_D^{(k)}[a, b_1, \cdots, b_k; c; x_1, \cdots, x_k] = \frac{\Gamma(c)}{\Gamma(a) \prod_{j=1}^{k} \Gamma(b_j)} \left( \frac{1}{2\pi \alpha} \right)^k \int_{L_1} \cdots \int_{L_k} \frac{\Gamma\left( a + \sum_{j=1}^{k} \zeta_j \right) \Gamma(b_1 + \zeta_1), \cdots, \Gamma(b_k + \zeta_k)}{\Gamma\left( c + \sum_{j=1}^{k} \zeta_j \right)} \prod_{j=1}^{k} \Gamma\left( -\zeta_j \right) x_1^{\zeta_1} \cdots x_k^{\zeta_k} \quad (2.2)
\]

where max \(|\arg(-x_1)|, \cdots, |\arg(-x_k)|\) < \(\pi, c \neq 0, -1, -2, \cdots\).

We first establish the formula

\[
\int_{a}^{b} (t - a)^{\alpha - 1} (b - t)^{\beta - 1} \prod_{j=1}^{k} (f_j t + g_j)^{\sigma_j} \, dt = (b - a)^{\alpha + \beta - 1} B(\alpha, \beta) \prod_{j=1}^{k} (af_j + g_j)^{\sigma_j} \times F_D^{(k)}[\alpha, -\sigma_1, \cdots, -\sigma_k; \alpha + \beta; \frac{(b - a)f_1}{af_1 + g_1}, \cdots, \frac{(b - a)f_k}{af_k + g_k}] \quad (2.3)
\]
where $a, b \in \mathbb{R}(a < b), \alpha, \beta, f_i, g_i, \sigma_i \in \mathbb{C}, (i = 1, \cdots, k); \min(\text{Re}(\alpha), \text{Re}(\beta)) > 0$ and

$$\max_{1 \leq j \leq k} \left\{ \frac{\left| (b-a)f_j \right|}{a f_j + g_j} \right\} < 1$$

$F^{(k)}_D$ is a Lauricella's function of $k$-variables, see Srivastava et al ([5], page 60).

The formula (2.2) can be establish by expanding $\prod_{j=1}^{k} (f_j t + g_j)^{\sigma_j}$ by means of the formula:

$$(1 - z)^{-\alpha} = \sum_{r=0}^{\infty} \frac{(\alpha)_r}{r!} z^r (|z| < 1)$$

(2.4)

integrating term by term with the help of the integral given by Saigo and Saxena [2, page 93, eq.(3.2)] and applying the definition of the Lauricella function $F^{(k)}_D$ [4, page 454].

3. Eulerian integral

Let

$$X = m'_1, n'_1; \cdots; m'_s, n'_s; m''_1, n''_1; \cdots; m''_u, n''_u; 1, 0; \cdots; 1, 0; 1, 0; \cdots; 1, 0$$

(3.1)

$$Y = p'_1, q'_1; \cdots; p'_s, q'_s; p''_1, q''_1; \cdots; p''_u, q''_u; 0, 1; \cdots; 0, 1; 0, 1; \cdots; 0, 1$$

(3.2)

$$A = (a'_j, A'_j^{(s)}, 0, \cdots, 0, 0, \cdots, 0, 0, \cdots, 0)_{1,p'}$$

(3.3)

$$B = (b'_j, B'_j^{(s)}, 0, \cdots, 0, 0, \cdots, 0, 0, \cdots, 0)_{1,q'}$$

(3.4)

$$A' = (a''_j; 0, \cdots, 0, A''_j^{(s)}, 0, \cdots, 0, 0, \cdots, 0)_{1,p''}$$

(3.5)

$$B' = (b''_j; 0, \cdots, 0, B''_j^{(s)}, 0, \cdots, 0, 0, \cdots, 0)_{1,q''}$$

(3.6)

$$C = (c'_j^{(s)}, C'_j^{(s)})_{1,p'_j}; \cdots; (c'_j^{(s)}, C'_j^{(s)})_{1,p'_j}; (c''_j^{(s)}, C''_j^{(s)})_{1,p''_j}; \cdots; (c''_j^{(s)}, C''_j^{(s)})_{1,p''_j}$$

(1, 0); \cdots; (1, 0); (1, 0); \cdots; (1, 0)

(3.7)

$$D = (d'_j^{(s)}, D'_j^{(s)})_{1,q'_j}; \cdots; (d'_j^{(s)}, D'_j^{(s)})_{1,q'_j}; (d''_j^{(s)}, D''_j^{(s)})_{1,q''_j}; \cdots; (d''_j^{(s)}, D''_j^{(s)})_{1,q''_j}$$

(0, 1); \cdots; (0, 1); (0, 1); \cdots; (0, 1)

(3.8)

$$K_1 = (1 - \alpha - \sum_{i=1}^{v} \eta_{G_i, g_i}(\mu_i + \mu'_i); \mu_1, \cdots, \mu_s, \mu'_1, \cdots, \mu'_u, 1, \cdots, 1, v_1, \cdots, v_l)$$

(3.9)
We have the following result

\[ K_2 = (1 - \beta - \sum_{i=1}^{\nu} \eta_{G_i, g_i}(\rho_i + \rho'_i); \rho_1, \cdots, \rho_s, \rho'_1, \cdots, \rho'_u, 0, \cdots, 0, \tau_1, \cdots, \tau_l) \]  

\[ K_P = [1 - A_j; 0, \cdots, 0, 0, \cdots, 0, 0, \cdots, 0, 1, \cdots, 1]_{1,P} \]  

\[ K_j = [1 + \sigma_j - \sum_{i=1}^{\nu} \eta_{G_i, g_i}(\lambda_i^{(j)} + \lambda'_i^{(j)}); \lambda_j^{(1)}, \cdots, \lambda_j^{(s)}, \lambda_j^{(u)}, 0, \cdots, 1, \cdots, 0, \xi_j, \cdots, \xi_j^{(l)}]_{1,k} \]  

\[ L_1 = (1 - \alpha - \beta - \sum_{i=1}^{\nu} \eta_{G_i, g_i}(\mu_i + \mu'_i + \rho_i + \rho'_i); \mu_1 + \rho_1, \cdots, \mu_s + \rho_s, \mu'_1 + \rho'_1, \cdots, \mu'_u + \rho'_u, \]  

\[ 1, \cdots, 1, \nu_1 + \tau_1, \cdots, \nu_l + \tau_l) \]  

\[ L_Q = [1 - B_j; 0, \cdots, 0, 0, \cdots, 0, 0, \cdots, 0, 1 \cdots, 1]_{1,Q} \]  

\[ L_j = [1 + \sigma_j - \sum_{i=1}^{\nu} \eta_{G_i, g_i}(\lambda_i^{(j)} + \lambda'_i^{(j)}); \lambda_j^{(1)}, \cdots, \lambda_j^{(s)}, \lambda_j^{(u)}, 0, \cdots, 0, \xi_j, \cdots, \xi_j^{(l)}]_{1,k} \]  

\[ A_1 = A, A'; B_1 = B, B' \]  

\[
\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1} \prod_{j=1}^k (f_j t + g_j)^{\sigma_j} dt \begin{pmatrix}
(z_1(t-a)^{\mu_1}(b-t)^{\rho_1} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j^{(1)}} \\
\vdots \\
z_u(t-a)^{\mu_u}(b-t)^{\rho_u} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j^{(u)}}
\end{pmatrix}
\]

\[
A \begin{pmatrix}
z_1(t-a)^{\mu'_1}(b-t)^{\rho'_1} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j^{(1)}} \\
\vdots \\
z_u(t-a)^{\mu'_u}(b-t)^{\rho'_u} \prod_{j=1}^k (f_j t + g_j)^{-\lambda_j^{(u)}}
\end{pmatrix}
\]
We obtain the A-function of variables.

\[ \text{P}_{FQ} \left[ (A_P; (B_Q) ; - \sum_{i=1}^{l} z_i''(t - a)^{\nu_i} (b - t)^{\tau_i} \prod_{j=1}^{k} (f_j t + g_j)^{- \zeta_j^{(i)}} \right] \, dt = (b - a)^{\alpha + \beta - 1} \]

\[ = \prod_{j=1}^{P} \frac{\prod_{j=1}^{Q} \Gamma(B_j)}{\prod_{j=1}^{M} \Gamma(A_j) \prod_{h_1=1}^{M_1} \cdots \prod_{h_u=1}^{M_u} \sum_{k_1=0}^{\infty} \cdots \sum_{k_u=0}^{\infty} \prod_{k_1=1}^{h_1} R_1 \cdots R_u \leq L_v \prod_{i=1}^{u} z_i'' \prod_{i=1}^{u} z_i'' \prod_{k=1}^{u} z_i'' \prod_{u=1}^{u} z_i'' R_k B_u B_{u,v} \]

\[ \begin{align*}
\frac{z_1 (b-a)^{\mu_1 + \rho_1}}{\prod_{j=1}^{k} (a f_j + g_j)^{\lambda_j^{(1)}}} \\
\frac{z_2 (b-a)^{\mu_2 + \rho_2}}{\prod_{j=1}^{k} (a f_j + g_j)^{\lambda_j^{(2)}}} \\
\vdots \\
\frac{z_s (b-a)^{\mu_s + \rho_s}}{\prod_{j=1}^{k} (a f_j + g_j)^{\lambda_j^{(s)}}} \\
\frac{z'_1 (b-a)^{\mu'_1 + \rho'_1}}{\prod_{j=1}^{k} (a f_j + g_j)^{\lambda'_j^{(1)}}} \\
\frac{z'_2 (b-a)^{\mu'_2 + \rho'_2}}{\prod_{j=1}^{k} (a f_j + g_j)^{\lambda'_j^{(2)}}} \\
\vdots \\
\frac{z'_u (b-a)^{\mu'_u + \rho'_u}}{\prod_{j=1}^{k} (a f_j + g_j)^{\lambda'_j^{(u)}}} \\
\frac{(b-a) f_1}{a f_1 + g_1} \\
\frac{(b-a) f_2}{a f_2 + g_2} \\
\vdots \\
\frac{(b-a) f_k}{a f_k + g_k} \\
\frac{z''_1 (b-a)^{\tau_1 + \nu_1}}{\prod_{j=1}^{k} (a f_j + g_j)^{\zeta_j^{(1)}}} \\
\frac{z''_2 (b-a)^{\tau_2 + \nu_2}}{\prod_{j=1}^{k} (a f_j + g_j)^{\zeta_j^{(2)}}} \\
\vdots \\
\frac{z''_u (b-a)^{\tau_u + \nu_u}}{\prod_{j=1}^{k} (a f_j + g_j)^{\zeta_j^{(u)}}}
\end{align*} \]

(A_1, K_1, K_2, K_P, K_j : C)

\( A^{m'+m''+l+k+2}; X^{p'+p''+l+k+2}; Q^{q'+q''+l+k+2}; Y \)

\( B_1, L_1, L_j, L_Q, : D \)

We obtain the A-function of \( s + u + k + l \) variables.

Where \( P_1 = (b-a)^{\alpha + \beta - 1} \left\{ \prod_{j=1}^{h} (a f_j + g_j)^{\sigma_j} \right\} \) \hspace{1cm} (3.18)

\( B_{u,v} = (b-a)^{\sum_{i=1}^{e} (\mu_i + \lambda_i + \rho_i + \rho'_i) \eta_i v_i} \left\{ \prod_{j=1}^{h} (a f_j + g_j)^{- \sum_{i=1}^{e} (\lambda_i + \lambda'_i) \eta_i v_i} \right\} G_v \) \hspace{1cm} (3.19)
where $G_v = \psi(\eta G_1, g_1, \cdots, \eta G_v, g_v) \times \xi_1(\eta G_1, g_1) \cdots \xi_v(\eta G_v, g_v)$

$\psi_1, \xi_i, i = 1, \cdots, v$ are defined respectively by (1.2) and (1.3)

$$B_u = \frac{(L)^{h_1 R_1 + \cdots + h_a R_a} B(E; R_1, \cdots, R_u)}{R_1! \cdots R_u!}$$ (3.20)

Provided that

(A) $m', n', p', m_i, n_i, p_i, c_i \in \mathbb{N}^n; i = 1, \cdots, s$ ; $a_j', b_j', c_j', d_j'(i), A_j'(i), B_j'(i), C_j'(i), D_j'(i) \in \mathbb{C}$

$\eta^{''}, \eta^{'''}, p^{''}, m^{''}, n^{''}, p^{''}, c^{''}, c^{'''} \in \mathbb{N}^n; i = 1, \cdots, u$ ; $a_j^{''}, b_j^{''}, c_j^{''}, d_j^{''}(i), A_j^{''}(i), B_j^{''}(i), C_j^{''}(i), D_j^{''}(i) \in \mathbb{C}$

(B) (A) $a, b \in \mathbb{R}(a < b); \mu_i, \rho_i, \mu_j, \rho_j \lambda_j(i), \lambda_j(i) \in \mathbb{R}^+$, $f_i, g_j, r_1, s_j \in \mathbb{C}$ ($i = 1, \cdots, r; j = 1, \cdots; s, u = 1, \cdots, k$)

$$\max_{1 \leq j \leq k} \left\{ \frac{(b - a) f_i}{\alpha f_i + g_i} \right\} < 1$$

(D) $Re\left[\alpha + \sum_{i=1}^{s} \mu_i \min_{1 \leq j \leq m_i} \frac{d_j'(i)}{D_j'(i)} + \sum_{i=1}^{s} \mu_i' \min_{1 \leq j \leq m_i''} \frac{d_j''(i)}{D_j''(i)}\right] > 0$

$$Re\left[\beta + \sum_{i=1}^{s} \rho_i \min_{1 \leq j \leq m_i} \frac{d_j'(i)}{D_j'(i)} + \sum_{i=1}^{s} \rho_i' \min_{1 \leq j \leq m_i''} \frac{d_j''(i)}{D_j''(i)}\right] > 0$

(E) $\xi_i = Im\left(\sum_{j=1}^{s} A_j'(i) - \sum_{j=1}^{s} B_j'(i) + \sum_{j=1}^{s} C_j'(i) - \sum_{j=1}^{s} D_j'(i)\right) = 0; i = 1, \cdots, s$

$\xi_i'' = Im\left(\sum_{j=1}^{u} A_j''(i) - \sum_{j=1}^{u} B_j''(i) + \sum_{j=1}^{u} C_j''(i) - \sum_{j=1}^{u} D_j''(i)\right) = 0; i = 1, \cdots, u$

(F) $|arg(\Omega_i)z_k| < \frac{1}{2} \eta \pi, \xi'' = 0, \eta > 0$

$$Re\left(\sum_{j=1}^{n} A_j'(i) - \sum_{j=n+1}^{p} A_j'(i) + \sum_{j=1}^{m} B_j'(i) - \sum_{j=m+1}^{q} B_j'(i) + \sum_{j=1}^{m_1} D_j'(i) - \sum_{j=m_1+1}^{q_1} D_j'(i) + \sum_{j=1}^{m} C_j'(i) - \sum_{j=m+1}^{q} C_j'(i)\right)$$

$- \mu_i' - \rho_i' - \sum_{i=1}^{k} \lambda_j'(i) > 0; i = 1, \cdots, s$

$$|arg(\Omega_i)z_k'| < \frac{1}{2} \eta_0 \pi, \xi'' = 0, \eta_0 > 0$$

$$Re\left(\sum_{j=1}^{n''} A_j''(i) - \sum_{j=n''+1}^{p''} A_j''(i) + \sum_{j=1}^{m''} B_j''(i) - \sum_{j=m''+1}^{q''} B_j''(i) + \sum_{j=1}^{m} D_j''(i) - \sum_{j=m+1}^{q} D_j''(i) + \sum_{j=1}^{m''} C_j''(i) - \sum_{j=m''+1}^{q''} C_j''(i)\right)$$
\[ -\mu'_t - \rho'_t - \sum_{i=1}^{k} \lambda^{(i)}_j > 0 \quad ; \quad i = 1, \ldots, u \]

\[
\text{G} \quad \arg \left( z_i \prod_{j=1}^{k} (f_j t + g_j)^{-\lambda^{(i)}_j} \right) < \frac{1}{2} \eta_i \pi \quad (a \leq t \leq b; i = 1, \ldots, s)
\]

\[
\arg \left( z'_i \prod_{j=1}^{k} (f_j t + g_j)^{-\lambda^{(i)}_j} \right) < \frac{1}{2} \eta'_i \pi \quad (a \leq t \leq b; i = 1, \ldots, u)
\]

\[
\text{H} \quad P \leq Q + 1. \text{ The equality holds, when , in addition,}
\]

either \( P > Q \) and \( z''_i \left( \prod_{j=1}^{k} (f_j t + g_j)^{-\lambda^{(i)}_j} \right) \mid_{t}^{b} < 1 \quad (a \leq t \leq b) \)

or \( P \leq Q \) and \( \max_{1 \leq i \leq k} \left[ \left( \prod_{j=1}^{k} (f_j t + g_j)^{-\lambda^{(i)}_j} \right) \right] < 1 \quad (a \leq t \leq b) \)

\[
\text{I} \quad \text{The multiple series occurring on the right-hand side of (3.17) is absolutely and uniformly convergent.}
\]

**Proof**

First expressing the multivariable Aleph-function in serie with the help of (1.6) and we interchange the order of summations and t-integral (which is permissible under the conditions stated). Expressing the A-function of s-variables and u-variables defined by Gautam et al [1] by the Mellin-Barnes contour integral with the help of the equation (1.9) and (1.17) respectively, the generalized hypergeometric function \( pF_q(.) \) in Mellin-Barnes contour integral with the help of (2.1) and interchange the order of integrations which is justifiable due to absolute convergence of the integral involved in the process. Now collect the power of \( (f_j t + g_j) \) with \( j = 1, \ldots, k \) and use the equations (2.1) and (2.2) and we obtain \( k \)-Mellin-Barnes contour integral. Interpreting \( (r + s + k + l) \)-Mellin-barnes contour integral in multivariable A-function defined by Gautam et al [1], we obtain the desired result.

4. Multivariable H-function

If \( A^{(i)}_t, B^{(i)}_t, C^{(i)}_t, D^{(i)}_t \in \mathbb{R}, n' = 0 \) and \( A''^{(i)}_t, B''^{(i)}_t, C''^{(i)}_t, D''^{(i)}_t \in \mathbb{R} \) and \( m'' = 0 \), the multivariable A-functions reduces to multivariable H-functions defined by Srivastava et al [6]. We obtain the following formula.

\[
\int_{a}^{b} (t-a)^{\alpha-1} (b-t)^{\beta-1} \prod_{j=1}^{k} (f_j t + g_j)^{\sigma_j} \left( z''_1 (t-a)^{\mu_1+\mu'_1} (b-t)^{\rho_1+\rho'_1} \prod_{j=1}^{k} (f_j t + g_j)^{-\lambda^{(1)}_j - \lambda^{(2)}_j} \right) \ldots \left( z''_v (t-a)^{\mu_v+\mu'_v} (b-t)^{\rho_v+\rho'_v} \prod_{j=1}^{k} (f_j t + g_j)^{-\lambda^{(v)}_j - \lambda^{(v+1)}_j} \right)
\]

\[
\begin{align*}
H & = \left( z_{1} (t-a)^{\mu_1} (b-t)^{\rho_1} \prod_{j=1}^{k} (f_j t + g_j)^{-\lambda^{(1)}_j} \right) \\
& \quad \ldots \left( z_{s} (t-a)^{\mu_s} (b-t)^{\rho_s} \prod_{j=1}^{k} (f_j t + g_j)^{-\lambda^{(s)}_j} \right)
\end{align*}
\]
\[
\begin{align*}
H & \left( z'_1(t-a)^{\mu'_1}(b-t)^{\nu'_1} \prod_{j=1}^{k}(f_j t + g_j)^{-\lambda_j} \right) \\
& \quad \vdots \\
& \left( z'_u(t-a)^{\mu'_u}(b-t)^{\nu'_u} \prod_{j=1}^{k}(f_j t + g_j)^{-\lambda_j'} \right)
\end{align*}
\]

\[
pFQ \left[ (A_P); (B_Q); -\sum_{i=1}^{l} z''_i(t-a)^{u_i}(b-t)^{v_i} \prod_{j=1}^{b}(f_j t + g_j)^{-\zeta_{ij}} \right] dt = (b-a)^{n+\beta-1}
\]

\[
= P_1 \prod_{p=1}^{Q} \frac{\Gamma(B_p)}{\Gamma(A_p)} \sum_{h_1=1}^{M_1} \cdots \sum_{h_u=1}^{M_u} \sum_{k=0}^{\infty} \cdots \sum_{k=0}^{\infty} \prod_{i=1}^{u} \prod_{k=1}^{u} \prod_{l=1}^{u} z'''_{h_1 \cdots h_u \cdots k_i \cdots k_u} B_{u_i} B_{u_i}
\]

\[
\begin{align*}
& \frac{z_1(b-a)^{\mu_1+\rho_1}}{\Pi_{j=1}^{k}(af_j+g_j)^{\lambda_1} j} \\
& \frac{z_2(b-a)^{\mu_2+\rho_2}}{\Pi_{j=1}^{k}(af_j+g_j)^{\lambda_2} j} \\
& \quad \vdots \\
& \frac{z_s(b-a)^{\mu_s+\rho_s}}{\Pi_{j=1}^{k}(af_j+g_j)^{\lambda_s} s} \\
& \frac{z''_1(b-a)^{\mu'_1+\rho'_1}}{\Pi_{j=1}^{k}(af_j+g_j)^{\lambda'_1} j} \\
& \frac{z''_2(b-a)^{\mu'_2+\rho'_2}}{\Pi_{j=1}^{k}(af_j+g_j)^{\lambda'_2} j} \\
& \quad \vdots \\
& \frac{z''_{u}(b-a)^{\mu'_u+\rho'_u}}{\Pi_{j=1}^{k}(af_j+g_j)^{\lambda'_u} u} \\
& \frac{(b-a) f_1}{af_1+g_1} \\
& \frac{(b-a) f_2}{af_2+g_2} \\
& \quad \vdots \\
& \frac{(b-a) f_k}{af_k+g_k} \\
& \frac{z''_1(b-a)^{\tau_1+v_1}}{\Pi_{j=1}^{k}(af_j+g_j)^{\zeta_{11}}} \\
& \frac{z''_2(b-a)^{\tau_1+v_1}}{\Pi_{j=1}^{k}(af_j+g_j)^{\zeta_{12}}} \\
& \quad \vdots \\
& \frac{z''_{u}(b-a)^{\tau_1+v_1}}{\Pi_{j=1}^{k}(af_j+g_j)^{\zeta_{1u}}} \\
& A_1, K_1, K_2, K_P, K_j : C \\
& B_1, L_1, L_j, L_Q : D
\end{align*}
\]

\[ H_0, n' + n'' + l + k + 2; X_{p' + p'' + l + k + 2, q' + q'' + l + k + 1}; Y \]
under the same conditions and notations that (3.17) with \( A_j^{(i)}, B_j^{(i)}, C_j^{(i)}, D_j^{(i)} \in \mathbb{R} \), \( m' = 0 \) and \( A_j^{(i)}, B_j^{(i)}, C_j^{(i)}, D_j^{(i)} \in \mathbb{R} \), \( m'' = 0 \).

**Remark**

By the following similar procedure, the results of this document can be extended to product of any finite number of multivariable A-functions defined by Gautam et al [1].

5. Conclusion

In this paper we have evaluated a generalized Eulerian integral involving the product of two multivariable A-functions, defined by Gautam et al [1], a expansion of multivariable Aleph-function and a generalized hypergeometric function with general arguments. The formula established in this paper is very general nature. Thus, the results established in this research work would serve as a key formula from which, upon specializing the parameters, as many as desired results involving the special functions of one and several variables can be obtained.

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