Second Hankel Determinant for Analytic Functions Defined By Linear Operator

Sunita M. Patil #1, S. M. Khairnar #2

Department Of Applied Sciences, Svps B.S. Deore College Of Engineering, Deopur, Dhaule, Maharashtra, India

Abstract: Let \( S(\lambda, n, m) \) denote the class of analytic and univalent functions in the open unit disk, \( D = \{ z : |z| < 1 \} \) with normalized conditions. In the present article an upper bound for the Second Hankel determinant \( |a_4a_3 - a_2^2| \) is obtained for the analytic functions defined by linear operator.

Key Words: Univalent function, Starlike function, convex function, Hankel determinant, Linear Operator.

1 INTRODUCTION, DEFINITION AND MOTIVATION

Let \( D \) be the unit disk \( \{ z : |z| < 1 \} \). \( A \) be the class of functions analytic in \( D \), satisfying the conditions,

\[
f(0) \text{ and } f'(0) = 1 \quad (1.1)
\]

then each function \( f \) in \( A \) has the Taylor's expansions,

\[
f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad Z \in D \quad (1.2)
\]

The \( q \)-th determinant for \( q \geq 1 \) and \( n \geq 1 \) is stated by Noonan and Thomas [14] as,

\[
H_q(n) = \begin{vmatrix}
a_{n_1} & a_{n_1+1} & \cdots & a_{n_1+q} \\
a_{n_2} & a_{n_2+1} & \cdots & a_{n_2+q} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n_q-1} & a_{n_q} & \cdots & a_{n_q+q-1}
\end{vmatrix} \quad (1.3)
\]

This determinant has also been considered by several authors. For example, Noonan [13] determinant the rate of growth of \( H_q(n) \) as \( n \to \infty \) for function \( f \) (1.1) with bounded boundary. Ebrenbary in [3] studied the Hankel determinant of exponential polynomials. The Hankel transform of an integer sequence and some of its properties were discussed by Layman S article[9]. It is well known that [4] for \( f \in S \) and given by (1.2) the sharp inequality \( |a_3a_2^2| \leq 1 \) holds. This corresponds to the Hankel determinant with \( q = 2 \) and \( k = 1 \). After that, Fekete-Szego further generalized the estimate \( |a_3 - \mu a_2^2| \) with real \( \mu \) and \( f \in S \) for a given class of functions in \( A \) the sharp bound for the non linear function \( |a_4a_3 - a_2^2| \) is known as the Second Hankel Determinant.

This corresponds to the Hankel determinant \( q = 2 \) and \( k = 2 \). In particular sharp bounds of article [8][12][16][17] for different subclass of univalent function,

\[
\begin{vmatrix}
a_2 & a_3 \\
a_3 & a_4
\end{vmatrix} = |a_2a_4 - a_3^2| \quad (1.4)
\]

Motivated by the above mentioned results obtained by different authors in this direction. In this paper we consider a certain subclass of analytic functions and obtain an upper bound to the function \( |a_2a_4 - a_3^2| \) for the function \( f \) belonging to this class defined as follows,

Definition 1.1 A function \( f \in A \) is said to be in the class,

\[
S^* = \{ f(z) \in S ; \Re \{ \frac{zf'(z)}{f(z)} \} > 0 \quad Z \in \{0\} \} \quad (1.5)
\]

\[
C = \{ f(z) \in S ; \Re \{ 1 + \frac{zf'(z)}{f(z)} \} > 0 \quad Z \in D \} \quad (1.6)
\]

for \( f_j \in A \) given by,

\[
f_j(z) = z + \sum_{k=1}^{n} a_k z^k \quad (j = 1,2) \quad (1.7)
\]

the Hadamard product (on convolution) \( f_1 \ast f_2 \) of \( f_1 \) and \( f_2 \) is defined by,

\[
(f_1 \ast f_2)z = z + \sum_{k=2}^{\infty} a_k a_{k-2} z^k \quad (Z \in D) \quad (1.8)
\]

recall that a family of the Hurwitz-Lerch zeta function \( \Phi_{\mu,\sigma}^{\alpha}(z,S,a) \) [13] is defined by,

\[
\Phi_{\mu,\sigma}^{\alpha}(z,S,a) = \sum_{n=0}^{\infty} (\mu)^n a^n (n+z)^{\alpha} \quad (1.9)
\]

\( \mu \in \mathbb{C}, \nu \in \mathbb{C} \setminus \{ \mathbb{Z} \}; \rho, \sigma \in \mathbb{R}^+; \rho < \sigma \quad \text{and} \quad S, z \in \mathbb{C}; \rho = \sigma \quad \text{and} \quad S \in \mathbb{C} \quad \text{when} \quad |z| < 1; \rho = \sigma \quad \text{and} \quad \Re(s - \mu + \nu) > 1 \quad \text{when} \quad |z| = 1 \)

Contains as its special case not only the Hurwitz-Lerch Zeta function,
The power series of \( p \) given in (2.1) converges in \( \Delta \) in to function \( p \) if and only if the Toeplitz determinant

\[
D_n = \begin{vmatrix} 2 & C_1 & C_2 & \ldots & C_n \\ C-1 & 2 & C_1 & \ldots & C_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C-n & C-n+1 & \ldots & \ldots & 2 \end{vmatrix}
\]

Where, \( n=1,2,3,\ldots \) & \( C_k = \overline{C_k} \) \( \forall \) non-negative

They are strictly positive except for \( p_k > 0, t_k \) real and \( t_k \neq t_j \) for \( k \neq j \) in this case

\[
D_n > 0 \text{ for } n < m - 1 \text{ & } D_n = 0 \text{ for } n \geq m
\]

(1.3) **Main Result**

**Theorem 3.1**

\[
|a_k a_{k+1} - a_{k+2}^2| \leq \frac{m^2(m+1)^2}{3^n(\lambda+1)^2(\lambda+2)^2}
\]

**Proof.** \( f \in S(\lambda, \alpha, n, m) \) there exist on analytic function \( p \in P \) in the unit disk \( D \) with \( p(0) = 1 \) and

\[
\lim_{n \to \infty} p_n(z) > 0,
\]

such that

\[
z \left[ \frac{L_{m,n,f}(z)}{L_{m,n,f}(z)} \right] = p(z) = 1 + p_1 z + p_2 z^2 + \ldots
\]

Equating the Coefficients,

\[
a_k = \frac{p_m}{(1+\lambda)^n}
\]

\[
a_j = \frac{(p_3 + p_4^2)m(m+1)}{2 \times 3^n(\lambda+1)(\lambda+2)}
\]

\[
a_k = \frac{(2p_1 + 3p_2 + p_3^2)m(m+1)(m+2)}{3 \times 4^n(\lambda+1)(\lambda+2)^2(\lambda+3)}
\]

using it we get,

\[
|a_k a_{k+1} - a_{k+2}^2| = A(m, \lambda) |2p_1 p_3 + 3p_2^2 p_3 + p_4 - \beta(m, \lambda) (p_2 + p_4) p_3^2|
\]

where,

\[
A(m, \lambda) = \frac{m^2(m+1)(m+2)}{3 \times 2^n \times 4^n(\lambda+1)^2(\lambda+2)(\lambda+3)}
\]

\[
\beta(m, \lambda) = \frac{(m+1)(\lambda+3) \times 3 \times 2^n \times 4^n}{4 \times 3^{2n}(\lambda+2)(m+2)}
\]
by applying Lemma (2.2) and (2.3),

$$= A(m, \lambda) \left[ \left( \frac{1}{2} \beta(m, \lambda) \right) p_1 + \left( 3 - 2 \beta(m, \lambda) \right) p_2 + 2p_1p_3 - \beta(m, \lambda) p_2 \right] \tag{3.8}$$

which also stated in Janteng et al.

**REFERENCES**


