Face Integer Cordial Labeling of Graphs

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Abstract - In this paper, we have introduced and investigated the face integer cordial labeling of wheel Wn fan fα, triangular snake Tn, double triangular snake DTn, star of cycle Cn and DS(Bn,n).

Keywords - Integer cordial labeling, face integer cordial labeling, face integer cordial graph.

I. INTRODUCTION

We begin with simple, finite, planar, undirected graph. A (p,q) planar graph G means a graph G(V,E), where V is the set of vertices with |V| = p, E is the set of edges with |E| = q and F is the set of interior faces of G with |F| = number of interior faces of G. For standard terminology and notations related to graph theory we refer to Harary [3]. A graph labeling is the assignment of unique identifiers to the edges and vertices of a graph. Graph labelings have enormous applications within mathematics as well as to several areas of computer science and communication networks. For a dynamic survey on various graph labeling problems along with an extensive bibliography we refer to Gallian [2].

A mapping f : V(G) → {0,1} is called binary vertex labeling of G and f(v) is called the label of the vertex v of G under f. If for an edge e = uv, the induced edge labeling f* : E(G) → {0,1} is given by f*(e) = |f(u)−f(v)|. Then v(i) is number of vertices having label i under f and e(i) is number of edges having label i under f*. A binary vertex labeling f of a graph G is called a cordial labeling of G if |v0(0) − v1(1)| ≤ 1 and |e0(0) − e1(1)| ≤ 1. A graph G is cordial if it admits cordial labeling. In [1], Cahit introduced the concept of cordial labeling of graph.

A product cordial labeling of a graph G with vertex set V is a function f from V to {0,1} such that if each edge uv is assigned a label f(u)f(v) then (i) the number of vertices labeled with 0 and the number of vertices labeled with 1 differ by at most 1 and (ii) the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. A graph with a product cordial labeling is called a product cordial graph. The concept of product cordial labeling of a graph was introduced by Sundaram et al. [8].

For graph G, the edge labeling function is defined as f : E(G) → {0,1} and induced vertex labeling function f* : V(G) → {0,1} is given as if e1,e2,...,en are the edges incident to vertex v then f*(v) = f(e1)f(e2)...f(en). Let us denote v(i) is the number of vertices of G having label i under f and e(i) is the number of edges of G having label i under f for i = 0,1. f is called edge product cordial labeling of graph G if |v0(0) − v1(1)| ≤ 1 and |e0(0) − e1(1)| ≤ 1. A graph G is called edge product cordial if it admits edge product cordial labeling. In [9], Vaidya et al. introduced the concept of edge product cordial labeling of graph.

Let a and b be two integers. If a divides b means that there is a positive integer k such that b = ak. It is denoted by ab. If a does not divide b, then we denote a|b. Let G = (V(G), E(G)) be a simple graph and f : V(G)→{1,2,...,|V(G)|} be a bijection. For each edge uv, assign the label 1 if f(u)f(v) or f(v)f(u) and the label 0 otherwise. The function f is called a divisor cordial labeling if |v0(0) − v1(1)| ≤ 1. A graph with a divisor cordial labeling is called a divisor cordial graph. Varatharajan et al. [10] introduced the concept of divisor cordial labeling of graphs.

For a planar graph G, the vertex labeling function is defined as g : V(G)→{0,1} and g(v) is called the label of the vertex v of G under g, induced edge labeling function g* : E(G)→{0,1} is given as if e = uv then g*(e) = g(u)g(v) and induced face labeling function g** : F(G)→{0,1} is given as if v1,v2,...,vn and e1,e2,...,en are the vertices and edges of face f, then g**(f) = g(v1)g(v2)...g(vn)g(e1)g(e2)...g(en). g(0) is the number of vertices of G having label i under g, e(i) is the number of edges of G having label i under g** for i = 0,1. g is called face product cordial labeling of graph G if |v0(0)−v1(1)| ≤ 1, |e0(0)−e1(1)| ≤ 1 and |f0(0)−f1(1)| ≤ 1. A graph G is face product cordial if it admits face product cordial labeling. Lawrence et al. introduced the concept of face product cordial labeling of graphs in [5] and they proved fan, M(Pn), S(Pn) except for odd n, T(Pn), Tn, Hn, Sn except for even n and one vertex union of mCn and nCn are face product cordial graph.

For a planar graph G, the edge labeling function is defined as g : E(G)→{0,1} and g(e) is called the label of the edge e of G under g, induced vertex labeling function g* : V(G)→{0,1} is given as if e1,e2,...,en are the edges incident to vertex v, then
A vertex labeling function is $g: V(G) \to \{0, 1\}$ is given as if $v_1, v_2, \ldots, v_n$ and $e_1, e_2, \ldots, e_n$ are the vertices and edges of face $f$ then $g^*(f) = g(v_1)g(v_2)\ldots g(v_n)g(e_1)g(e_2)\ldots g(e_n)$. $v_g(i)$ is the number of vertices of $G$ having label $i$ under $g^*$, $e_g(i)$ is the number of edges of $G$ having label $i$ under $g^*$ and $f_g(i)$ is the number of interior faces of $G$ having label $i$ under $g^*$ for $i = 1, 2$. A planar graph $G$ is face integer cordial if it admits face integer cordial labeling.

For a planar graph $G$, an edge labeling function is defined as $g: E \to [-\frac{p}{2}, \frac{p}{2}]$ or $[-\frac{p}{2}, \ldots, \frac{p}{2}]$ as $p$ is even or odd be an injective map, which induces vertex labeling function $g^*: V(G) \to \{0, 1\}$ such that $g^*(v) = 1$, if $\sum g(e_i) \geq 0$ and $g^*(v) = 0$ otherwise, where $e_1, e_2, \ldots, e_n$ are the adjacent edges of the vertex $v$ and face labeling function $g^**: F(G) \to \{0, 1\}$ such that $g^**(f) = 1$, if $g^*(f) = g(e_1) + g(e_2) + \ldots + g(e_n) \geq 0$ and $g^**(f) = 0$ otherwise, where $e_1, e_2, \ldots, e_n$ are the edges of face $f$.

Let $G$ be a simple connected graph with $p$ vertices. Let $f: V \to [-\frac{p}{2}, \frac{p}{2}]$ or $[-\frac{p}{2}, \ldots, \frac{p}{2}]$ as $p$ is even or odd be an injective map, which induces an edge labeling $f^*$ such that $f(uv) = 1$, if $f(u)+f(v) \geq 0$ and $f(uv) = 0$ otherwise. Let $e(f)$ be number of edges labeled with $i$, where $i = 0$ or $1$. $f$ is said to be integer cordial if $|e(0)−e(1)| \leq 1$. A graph $G$ is called integer cordial if it admits an integer cordial labeling. Here $[−x, \ldots, x] = \{t / t \text{ is an integer} \text{ and } |t| \leq x\}$ and $[−x, \ldots, x] = \{−x, \ldots, x\}−\{0\}$.

In [7], Nicholas et al. introduced the concept of integer cordial labeling of graphs and proved that some standard graphs such as cycle $C_n$. Path $P_n$. Wheel graph $W_n$. $n \geq 3$. Star graph $K_{1,n}$. Helm graph $H_n$. Closed helm graph $CH_n$. Wheel $W_n$. $n \geq 3$. Star graph $K_{1,n}$. Helm graph $H_n$. Closed helm graph $CH_n$ are integer cordial, $K_n$ is not integer cordial, $K_{1,n}$ is integer cordial iff $n$ is even and $K_{n,n}$ is integer cordial for any $n$, where $M$ is a perfect matching of $K_{n,n}$.

Motivated by the concept of face product cordial labeling, face edge product cordial labeling and integer cordial labeling, we introduce two new types of labeling such as face integer cordial and face integer cordial edge labeling of graph. For a planar graph $G$, the vertex labeling function is defined as $g: V \to [-\frac{p}{2}, \frac{p}{2}]$ or $[-\frac{p}{2}, \ldots, \frac{p}{2}]$ as $p$ is even or odd be an injective map, which induces an edge labeling function $g^*: E(G) \to \{0, 1\}$ such that $g^*(uv) = 1$, if $g(u)+g(v) \geq 0$ and $g^*(uv) = 0$ otherwise and face labeling function $g^{**}: F(G) \to \{0, 1\}$ such that $g^{**}(f) = 1$, if $g^*(f) = g(v_1) + g(v_2) + \ldots + g(v_n) \geq 0$ and $g^{**}(f) = 0$ otherwise, where $v_1, v_2, \ldots, v_n$ are the vertices of face $f$. $g$ is called face integer cordial labeling of graph $G$ if $|e_g(0)−e_g(1)| \leq 1$ and $|f_g(0)−f_g(1)| \leq 1$. $e_g(i)$ is the number of edges of $G$ having label $i$ and $f_g(i)$ is the number of interior faces of $G$ having label $i$ for $i = 1, 2$. A planar graph $G$ is face integer cordial if it admits face integer cordial labeling.

In [6], Mohamed Sherif et al proved wheel graph, fan graph, friendship graph, triangular snake, alternative triangular snake, star of cycle, degree splitting graph of bistar, vertex switching of cycle, pendant vertex switching of path, helm, closed helm, middle graph of path and total graph of path are face integer edge cordial graph.

The present work is focused only on face integer cordial labeling of some new families of graphs. The face integer cordial labeling of wheel $W_n$, fan $F_n$, triangular snake $T_n$, double triangular snake $DT_n$, star of cycle $C_n$ and DS$(B_n)$ is presented. The brief summaries of definition which are necessary for the present investigation are provided below.

Definition : 1.1

A wheel $W_n$ is a graph with $n+1$ vertices, formed by connecting a single vertex to all the vertices of cycle $C_n$. It is denoted by $W_n = C_n + K_1$.

Definition : 1.2

A triangular snake $T_n$ is obtained from a path $u_1, u_2, \ldots, u_n$ by joining $u_i$ and $u_{i+1}$ to a new vertex $v_i$ for $i = 1, 2, \ldots, n−1$.

Definition : 1.3

The friendship graph $F_n$ is one-point union of $n$ copies of cycles $C_3$.

Definition : 1.4

The join of two graphs $G$ and $H$ is a graph $G \cup H$ with $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$ or $\{uv : u \in V(G) \text{ and } v \in V(H)\}$. The graph $P_n + K_1$ is called a fan of $n$ vertices and is denoted by $f_n$. 
Definition 1.5

Let G be a graph with two or more vertices than the total graph T(G) of graph G is the graph whose vertex set is V(G)∪E(G) and two vertices are adjacent whenever they are either adjacent or incident in G.

Definition 1.6

Let G be a graph with vertex set V = S1∪S2∪…∪Sk/T where each Si is a set of vertices having at least two vertices of the same degree and T = V\cup S. The degree splitting graph of G denoted by DS(G) is obtained from G by adding vertices wi, w2, w3, …, wn and joining to each vertex of Si for 1 ≤ i ≤ n.

Remark 1.1

Any unicyclic integer cordial graphs are face integer cordial graphs.

Remark 1.2

Every planar graph G is always a subgraph of the face integer cordial graph G∪G.

II. MAIN THEOREMS

Theorem 2.1

The wheel Sn is a face integer cordial graph for n ≥ 3.

Proof

Let v be the apex vertex, v1, v2, …, vn be rim vertices, e1, e2, …, en be edges and f1, f2, …, fn be interior faces of the wheel Sn where e1 = vv, for i = 1, 2, …, n, ei = vi+1v, for i = 1, 2, …, n−1, e2n = vn+1v, f1 = vv, f2 = vi+1v, for i = 1, 2, …, n−1 and fn = vn+1v.

Let G be the wheel graph Sn.

Then |V(G)| = n+1, |E(G)| = 2n and |F(G)| = n.

Case (i) n is odd.

Let n = 2k+1.

Define g : V(G) → [-k, …, k] as follows:

\[ g(v) = 1 \]
\[ g(v_i) = -i \quad \text{for } 1 \leq i \leq \frac{n+1}{2} \]
\[ g(v_{\frac{n+1}{2}}) = i+1 \quad \text{for } 1 \leq i \leq \frac{n-1}{2} \]

Then induced edge labels are

\[ g*(e_i) = 1 \]
\[ g*(e_i) = 0 \quad \text{for } 2 \leq i \leq \frac{n+1}{2} \]
\[ g*(e_i) = 1 \quad \text{for } n+3 \leq i \leq n \]
\[ g*(e_{n+1}) = 0 \quad \text{for } 1 \leq i \leq \frac{n+1}{2} \]
\[ g*(e_{n+1}) = 1 \quad \text{for } n+3 \leq i \leq n \]

Also the induced face labels are

\[ g**(f_i) = 0 \quad \text{for } 1 \leq i \leq \frac{n-1}{2} \]
\[ g**(f_i) = 1 \quad \text{for } \frac{n+1}{2} \leq i \leq n \]

In view of the above defined labeling pattern, we have

\[ e_i(0) = e_i(1) = n \quad \text{and} \quad f_i(1) = f_i(0) + 1 = \frac{n+1}{2}. \]

Then |e_i(0) − e_i(1)| ≤ 1 and |f_i(0) − f_i(1)| ≤ 1.

Thus the wheel Sn is the face integer cordial for n is odd.

Case 2: n is even.

Let n = 2k.

Define g : V(G) → [-k, …, k] as follows:

\[ g(v) = 0 \]
\[ g(v_i) = -i \quad \text{for } 1 \leq i \leq \frac{n}{2} \]
\[ g(v_{\frac{n}{2}}) = i \quad \text{for } 1 \leq i \leq \frac{n}{2} \]

Then induced edge labels are

\[ g*(e_i) = 0 \quad \text{for } 1 \leq i \leq \frac{n}{2} \]
\[ g*(e_i) = 1 \quad \text{for } \frac{n+2}{2} \leq i \leq n \]
\[ g*(e_{n+1}) = 0 \quad \text{for } 1 \leq i \leq \frac{n}{2} \]
\[ g*(e_{n+1}) = 1 \quad \text{for } \frac{n+2}{2} \leq i \leq n \]

Also the induced face labels are

\[ g**(f_i) = 0 \quad \text{for } 1 \leq i \leq \frac{n}{2} \]
\[ g**(f_i) = 1 \quad \text{for } \frac{n+2}{2} \leq i \leq n \]

In view of the above defined labeling pattern, we have

\[ e_i(0) = e_i(1) = n \quad \text{and} \quad f_i(1) = f_i(0) + 1 = n. \]

Then |e_i(0) − e_i(1)| ≤ 1 and |f_i(0) − f_i(1)| ≤ 1.

Thus the wheel Sn is the face integer cordial for n is even.

Hence the wheel Sn is the face integer cordial graph for n ≥ 3.

Example 2.1

The wheel S3 and its face integer cordial labeling is shown in figure 2.1.

![Figure 2.1](http://www.iijmttjournal.org)
Theorem 2.2

The fan $f_n$ is face integer cordial graph for $n \geq 2$.

Proof:

Let $v_1,v_2,\ldots,v_n, e_1,e_2,\ldots,e_{2n-1}$ and $f_1,f_2,\ldots,f_{n-1}$ be the vertices, edges and an interior faces of $f_n$, where $e_i=v_{i+1}$ for $i=1,2,\ldots,n$ and $e_{n+i}=v_{n+i+1}$ for $i=1,2,\ldots,n-1$.

Let $G$ be the fan graph $f_n$. Then $|V(G)| = n+1$, $|E(G)| = 2n-1$ and $|F(G)| = n-1$.

Case (i): $n$ is odd and $n = 2k+1$.

Define $g : V(G) \rightarrow \{-(k+1),\ldots,(k+1)\}$ as follows.

$$g(v_i) = 1 + i \quad \text{for} \quad 1 \leq i \leq \frac{n-1}{2}$$

$$g(v_{n+i}) = -i \quad \text{for} \quad 1 \leq i \leq \frac{n+1}{2}$$

Then induced edge labels are

$$g^*(e_i) = 1 \quad \text{for} \quad 1 \leq i \leq \frac{n+1}{2}$$

$$g^*(e_i) = 0 \quad \text{for} \quad \frac{n+3}{2} \leq i \leq n$$

$$g^*(e_{n+i}) = 1 \quad \text{for} \quad 1 \leq i \leq \frac{n-1}{2}$$

$$g^*(e_{n+i}) = 0 \quad \text{for} \quad \frac{n+1}{2} \leq i \leq n-1$$

Also the induced face labels are

$$g^{**}(f_i) = 1 \quad \text{for} \quad 1 \leq i \leq \frac{n}{2}$$

$$g^{**}(f_i) = 0 \quad \text{for} \quad \frac{n+2}{2} \leq i \leq n-1$$

Proof:

$$g^{**}(f_i) = 0 \quad \text{for} \quad \frac{n+2}{2} \leq i \leq n-1$$

In view of the above defined labeling pattern, we have $e_i = e_i(0)+1 = n$ and $f_i = f_i(0)+1 = \frac{n}{2}$.

Thus $|e_i(0) - e_i(1)| \leq 1$ and $|f_i(0) - f_i(1)| \leq 1$.

Therefore the fan $f_n$ is the face integer cordial graph for $n$ is odd.

Case (ii): $n$ is even and $n = 2k$.

Define $g : V(G) \rightarrow \{-k,\ldots,k\}$ as follows.

$$g(v_i) = 0$$

$$g(v_i) = i \quad \text{for} \quad 1 \leq i \leq \frac{n}{2}$$

Then induced edge labels are

$$g^*(e_i) = 1 \quad \text{for} \quad 1 \leq i \leq \frac{n}{2}$$

$$g^*(e_i) = 0 \quad \text{for} \quad \frac{n+1}{2} \leq i \leq n$$

$$g^*(e_{n+i}) = 1 \quad \text{for} \quad 1 \leq i \leq \frac{n}{2}$$

$$g^*(e_{n+i}) = 0 \quad \text{for} \quad \frac{n+1}{2} \leq i \leq n-1$$

Also the induced face labels are

$$g^{**}(f_i) = 1 \quad \text{for} \quad 1 \leq i \leq \frac{n}{2}$$

$$g^{**}(f_i) = 0 \quad \text{for} \quad \frac{n+1}{2} \leq i \leq n-1$$

Theorem 2.3

Triangular snake $T_n$ is face integer cordial graph for $n \geq 2$.

Proof:

Let $v_1,v_2,\ldots,v_n,u_1,u_2,\ldots,u_{n-1}$ be vertices, $e_1,e_2,\ldots,e_{3n-3}$ be edges and $f_1,f_2,\ldots,f_{n-1}$ interior faces of $T_n$, where $e_{2i-1} = v_i u_i$, $e_{2i} = u_i u_{i+1}$ and $e_{3n-3} = v_n V_{1}$ for $i = 1,2,\ldots,n-1$ and $f_i = v_i u_i V_{1}$ for $i = 1,2,\ldots,n-1$.

Let $G$ be the graph $T_n$. Then $|V(G)| = 2n-1$, $|E(G)| = 3n-3$ and $|F(G)| = n-1$.

Define $g : V(G) \rightarrow \{-n,\ldots,n\}$ as follows.

Case (i): $n$ is odd.

$$g(u_i) = i \quad \text{for} \quad 1 \leq i \leq \frac{n-1}{2}$$

$$g(u_i) = \frac{n-1}{2} - i \quad \text{for} \quad \frac{n+1}{2} \leq i \leq n-1$$

$$g(v_i) = \frac{n-1}{2} + i \quad \text{for} \quad 1 \leq i \leq \frac{n-1}{2}$$

$$g(v_i) = 0 \quad \text{for} \quad \frac{n+1}{2} \leq i \leq n$$

Then induced edge labels are

$$g^*(e_i) = 1 \quad \text{for} \quad 1 \leq i \leq n-1$$

$$g^*(e_i) = 0 \quad \text{for} \quad \frac{n+1}{2} \leq i \leq 2n-2$$

$$g^*(e_{3n-3}) = 1 \quad \text{for} \quad 1 \leq i \leq \frac{n-1}{2}$$

$$g^*(e_{3n-3}) = 0 \quad \text{for} \quad \frac{n+1}{2} \leq i \leq n-1$$

Also the induced face labels are

$$g^{**}(f_i) = 1 \quad \text{for} \quad 1 \leq i \leq \frac{n-1}{2}$$

$$g^{**}(f_i) = 0 \quad \text{for} \quad \frac{n+1}{2} \leq i \leq n-1$$

http://www.ijmttjournal.org
In view of the above defined labeling pattern we have $e_i(0) = e_i(1) = \frac{3n - 3}{2}$ and $f_i(0) = f_i(1) = \frac{n - 1}{2}$.

Then $|e_i(0) - e_i(1)| \leq 1$ and $|f_i(0) - f_i(1)| \leq 1$.

Thus $T_n$ is face integer cordial graph for $n$ is odd.

**Case (ii) :** $n$ is even.

- $g(u_i) = i$ for $1 \leq i \leq \frac{n - 2}{2}$
- $g(u_i) = 0$ for $i = \frac{n}{2}$
- $g(v_i) = -i + \frac{n}{2}$ for $\frac{n + 2}{2} \leq i \leq n - 1$
- $g(v_i) = i + \frac{n}{2}$ for $1 \leq i \leq \frac{n}{2}$
- $g(v_i) = -i - 1$ for $\frac{n + 2}{2} \leq i \leq n$

Then induced edge labels are

- $g^*(e_i) = 1$ for $1 \leq i \leq n - 1$
- $g^*(e_i) = 0$ for $n \leq i \leq 2n - 2$
- $g^*(e_{2n-2i}) = 1$ for $1 \leq i \leq \frac{n}{2}$
- $g^*(e_{2n-2i}) = 0$ for $\frac{n + 2}{2} \leq i \leq n - 1$

Also the induced face labels are

- $g^{**}(f_i) = 1$ for $1 \leq i \leq \frac{n}{2}$
- $g^{**}(f_i) = 0$ for $\frac{n + 2}{2} \leq i \leq n - 1$

In view of the above defined labeling pattern, we have

$e_i(1) = e_i(0) + 1 = \frac{3n - 2}{2}$ and $f_i(1) = f_i(0) + 1 = \frac{n + 1}{2}$.

Then $|e_i(0) - e_i(1)| \leq 1$ and $|f_i(0) - f_i(1)| \leq 1$.

Thus $T_n$ is face integer cordial graph for $n$ is even.

Hence $T_n$ is face integer cordial graph for $n \geq 2$.

**Example 2.3**

The graph $T_3$ and its face integer cordial labeling is shown in figure 2.3.

![Figure 2.3](image)

**Theorem : 2.4**

Double triangular snake $DT_n$ is a face integer cordial graph for $n \geq 3$.

**Proof.**

Let $v_0, v_1, v_2, \ldots, v_n$, $u_1, u_2, \ldots, u_{n-1}$, $w_1, w_2, \ldots, w_{n-1}$ be vertices, $e_1, e_2, \ldots, e_{3n-5}$ be edges and $f_1, f_2, \ldots, f_{3n-5}$ be an interior faces of $DT_n$, where $e_{2n-1} = v_i u_{n-1}$, $e_{2n} = u_{n-1} v_i$. $e_{2n+1} = v_i v_{i+1}$, $e_{3n+2i-4} = v_i w_i$, and $e_{3n+2i-3} = w_i v_i$ for $i = 1, 2, \ldots, n - 1$, $f_i = v_i w_{i+1} v_i$ for $i = 1, 2, \ldots, n - 1$ and $f_{n-1} = v_{i+1} w_i v_i$ for $i = 1, 2, \ldots, n - 1$.

Let $G$ be the double triangular snake $DT_n$. Then $|V(G)| = 3n - 2$, $|E(G)| = 5n - 5$ and $|F(G)| = 2n - 2$.

**Case (i) :** $n$ is odd and $k = \frac{3n - 3}{2}$.

Define $g : V(G) \rightarrow [-k, \ldots, k]$ as follows

- $g(u_i) = i$ for $1 \leq i \leq \frac{n - 1}{2}$
- $g(u_i) = \frac{n - 1}{2} - i$ for $\frac{n + 1}{2} \leq i \leq n - 1$
- $g(v_i) = \frac{n - 1}{2} + i$ for $1 \leq i \leq \frac{n - 1}{2}$
- $g(v_i) = 0$ for $i = \frac{n + 1}{2}$
- $g(v_i) = -i - 1$ for $\frac{n + 3}{2} \leq i \leq n$
- $g(w_i) = n - i + 1$ for $1 \leq i \leq \frac{n - 1}{2}$
- $g(w_i) = \left(\frac{n - 1}{2}\right) - i$ for $\frac{n + 1}{2} \leq i \leq n - 1$

Then induced edge labels are

- $g^*(e_i) = 1$ for $1 \leq i \leq n - 1$
- $g^*(e_i) = 0$ for $n \leq i \leq 2n - 2$
- $g^*(e_{2n-2i}) = 1$ for $1 \leq i \leq \frac{n}{2}$
- $g^*(e_{2n-2i}) = 0$ for $\frac{n + 2}{2} \leq i \leq n - 1$

Also the induced face labels are

- $g^{**}(f_i) = 1$ for $1 \leq i \leq \frac{n}{2}$
- $g^{**}(f_i) = 0$ for $\frac{n + 2}{2} \leq i \leq n - 1$

In view of the above defined labeling pattern, we have

$e_i(1) = e_i(0) + 1 = \frac{5n - 5}{2}$ and $f_i(1) = f_i(0) + 1 = -\frac{n - 1}{2}$.

Then $|e_i(0) - e_i(1)| \leq 1$ and $|f_i(0) - f_i(1)| \leq 1$.

Thus the graph $DT_n$ is face integer cordial graph for $n$ is odd.

**Case 2 :** $n$ is even and $k = \frac{3n - 2}{2}$.

Define $g : V(G) \rightarrow [-k, \ldots, k]$ as follows

- $g(u_i) = i - \frac{3n - 2}{2}$ for $1 \leq i \leq \frac{n - 2}{2}$
- $g(u_i) = \frac{n - 1}{2} - i$ for $\frac{n + 1}{2} \leq i \leq n - 1$
- $g(v_i) = \frac{n - 1}{2} + i$ for $1 \leq i \leq \frac{n - 1}{2}$
- $g(v_i) = 0$ for $i = \frac{n + 1}{2}$
- $g(v_i) = -i - 1$ for $\frac{n + 3}{2} \leq i \leq n$
- $g(w_i) = n - i + 1$ for $1 \leq i \leq \frac{n - 1}{2}$
- $g(w_i) = \left(\frac{n - 1}{2}\right) - i$ for $\frac{n + 1}{2} \leq i \leq n - 1$

In view of the above defined labeling pattern, we have

$e_i(0) = e_i(1) = \frac{5n - 5}{2}$ and $f_i(0) = f_i(1) = -\frac{n - 1}{2}$.

Then $|e_i(0) - e_i(1)| \leq 1$ and $|f_i(0) - f_i(1)| \leq 1$.

Thus the graph $DT_n$ is face integer cordial graph for $n$ is odd.
Theorem 2.5

The friendship graph $F_n$ is face integer cordial graph for $n \geq 3$.

Proof: Let $v_1, v_2, \ldots, v_{2n}$, $e_1, e_2, \ldots, e_{2n}$, $f_1, f_2, \ldots, f_n$ be the vertices, edges, and interior faces of $F_{2n}$, where $e_{2i-1} = v_{2i-1}v_{2i}$, $e_{2i} = v_{2i}v_{2i+1}$ and $f_i = v_{2i}v_{2i+1}v_{2i+2}$, for $1 \leq i \leq n$. Let $G$ be the friendship graph $F_{2n}$. Then $|V(G)| = 2n+1$, $|E(G)| = 3n$ and $|F(G)| = n$.

Define $g: V(G) \rightarrow [-n, \ldots, n]$ as follows

Case (i) : $n$ is odd

$g(v) = 0$ for $1 \leq i \leq n$
$g(v_1) = 1$ for $1 \leq i \leq n$
$g(v_{2n}) = -1$ for $1 \leq i \leq n$

Then induced edge labels are

$g^*(e_i) = 0$ for $1 \leq i \leq n-1$
$g^*(e_i) = 1$ for $1 \leq i \leq 2n-2$
$g^*(e_{2n-2i}) = 0$ for $1 \leq i \leq n-1$
$g^*(e_{2n-2i}) = 1$ for $1 \leq i \leq n-1$

Also the induced face labels are

$g^{**}(f_i) = 0$ for $1 \leq i \leq n-2$
$g^{**}(f_i) = 1$ for $1 \leq i \leq n-1$
$g^{**}(f_{n-1+2i}) = 0$ for $1 \leq i \leq n-1$
$g^{**}(f_{n-1+2i}) = 1$ for $1 \leq i \leq n-1$

In view of the above defined labeling pattern, we have $|e_i(0) - e_i(1)| = 1$ and $|f_i(0) - f_i(1)| = 1$.

Then $|e_i(0) - e_i(1)| \leq 1$ and $|f_i(0) - f_i(1)| \leq 1$.
Hence $F_n$ is face integer cordial graph.

Case (ii) : $n$ is even

$g(v) = 0$ for $1 \leq i \leq 2n$
$g(v_{2n}) = 1$ for $1 \leq i \leq n$
$g(v_{2n}) = 1$ for $1 \leq i \leq n$

Then induced edge labels are

$g^*(e_i) = 1$ for $1 \leq i \leq 3n-2$
$g^*(e_i) = 0$ for $1 \leq i \leq 3n$.

Also the induced face labels are

$g^{**}(f_i) = 1$ for $1 \leq i \leq n+1$
$g^{**}(f_i) = 0$ for $1 \leq i \leq 3n$.

In view of the above defined labeling pattern, we have $|e_i(0) - e_i(1)| = 1$ and $|f_i(0) - f_i(1)| = 1$.

Thus $|e_i(0) - e_i(1)| \leq 1$ and $|f_i(0) - f_i(1)| \leq 1$.
Hence $F_n$ is face integer cordial graph for $n$ is even.

Hence $F_n$ is face integer cordial graph for $n \geq 3$.
Example 2.5

The graph $F_1$ and its face integer cordial labeling is shown in figure 2.5.

![Figure 2.5](image-url)

Theorem 2.6

$DS(B_{n,n})$ is face integer cordial graph for $n \geq 2$.

**Proof.**

Let $u,v,u_1,u_2,\ldots,u_n,v_1,v_2,\ldots,v_n$ and $e_1,e_2,\ldots,e_{2n+1}$ be the vertices and edges of $B_{n,n}$.

Now $V(B_{n,n}) = V_1 \cup V_2$, where $V_1 = \{u,v\}$ and $V_2 = \{u_1,u_2,\ldots,u_n,v_1,v_2,\ldots,v_n\}$. In order to obtain $DS(B_{n,n})$ is obtained from $B_{n,n}$ by adding the vertex $w_1$ to $V_1$ and $w_2$ to $V_2$.

$u,v,u_1,u_2,\ldots,u_n,v_1,v_2,\ldots,v_n,w_1,w_2,e_1,e_2,\ldots,e_{2n+3}$ and $f_1,f_2,\ldots,f_2n$ be the vertices, edges and an interior faces of $DS(B_{n,n})$, where $e_i = uu_i$ for $i = 1,2,\ldots,n$, $e_{n+i} = uv$, $e_{n+i+1} = vv$, $e_{2n+1} = w_1u$, $e_{n+i+1} = w_1v$ for $i = 1,2,\ldots,n$, $e_{2n+2} = w_2u$, $e_{2n+3} = w_2v$ and $f_i = uu_iw_iu_i$, $f_{n+1} = vv$ for $i = 1,2,\ldots,n$, $f_{2n+1} = uuvw_iu_i$, and $f_{2n} = uuvw_iu_i$.

Let $G$ be a graph $DS(B_{n,n})$. Then $|V(G)| = 2n+4$, $|E(G)| = 4n+3$ and $|F(G)| = 2n$.

Define $g : V(G) \rightarrow \{-(n+2),\ldots,(n+2)\}$ as follows.

$f(u) = 2$

$f(v) = -1$

$f(w_1) = 2$

$f(w_2) = -(n+2)$

$f(u_i) = n+3 - i$ for $1 \leq i \leq n$

$f(v_i) = -(i+1)$ for $1 \leq i \leq n$.

Then induced edge labels are

$g^*(e_i) = 1$ for $1 \leq i \leq n$

$g^*(e_{n+i}) = 1$

$g^*(e_{n+i+1}) = 1$ for $1 \leq i \leq n$

$g^*(e_{2n+1}) = 1$

$g^*(e_{2n+1}) = 0$ for $1 \leq i \leq n+2$

Also the induced face labels are

$g^*(f_1) = 1$ for $1 \leq i \leq n-1$

$g^*(f_{n+1}) = 0$ for $1 \leq i \leq n+1$

In view of the above defined labeling pattern, we have $e_1(0) = e_1(1) = 2n+2$ and $f_1(1) = f_1(0) = n$.

Then $|e_1(0)-e_1(1)| \leq 1$ and $|f_1(0)-f_1(1)| \leq 1$.

Hence $DS(B_{n,n})$ is the face integer cordial for $n \geq 3$.

Example 2.6

The graph $DS(B_{1,3})$ and its face integer cordial labeling is shown in figure 2.6.

![Figure 2.6](image-url)

Theorem 2.7

The star of cycle $C_n$ is face integer cordial graph for $n \geq 3$.

**Proof.**

Let $v_1,v_2,\ldots,v_n, v_{11},v_{12},\ldots,v_{1n}, v_{21},v_{22},\ldots,v_{2n},\ldots,$ $v_{n1},v_{n2},\ldots,v_{nn},e_1,e_2,\ldots,e_{2n},e_{11},e_{12},\ldots,e_{1n},e_{21},e_{22},\ldots,e_{2n},\ldots,$ $e_{n1},e_{n2},\ldots,e_{nn}$ and $f_1, f_2, \ldots, f_{2n}$ be vertices, edges and an interior faces of the star of cycle $C_n$. $v_1,v_2,\ldots,v_n$ be the vertices of central cycle $C_n$, $v_{11},v_{12},\ldots,v_{1n}$ be the vertices of the cycle $C_1$, where $1 \leq i \leq n$ and $v_1$ is adjacent to the $i^{th}$ vertex of the central cycle $C_n$, $e_i = v_i v_{i+1}$, for $1 \leq i \leq n-1$, $e_n = v_n v_1$, for $1 \leq i \leq n$, $e_{i} = v_{i+1} v_{i+1}$, for $1 \leq i \leq n$ and $1 \leq j \leq n-1$, $e_{nn} = v_nv_1$, for $1 \leq i \leq n$, $f_1 = v_1 v_2 \ldots v_n$ and $f_{2n} = v_{11} v_{22} \ldots v_{nn}$ for $1 \leq i \leq n$.

Let $G$ be the star of cycle $C_n$.

Then $|V(G)| = n(n+1)$, $|E(G)| = n(n+2)$ and $|F(G)| = n+1$.

Case (i) : $n$ is even and $k = \frac{n(n+1)}{2}$

Define $g : V(G) \rightarrow \{-[i-1],\ldots,k\}$ as follows.

$g(v_i) = 1$, for $1 \leq i \leq n$

$g(v_i) = 1$, for $1 \leq i \leq n+1$

And $1 \leq j \leq n$.

Then induced edge labels are

$g^*(e_i) = 1$, for $1 \leq i \leq n+1$

$g^*(e_{nn}) = 0$, for $2 \leq i \leq \frac{n+1}{2}$

$g^*(e_{nn}) = 1$, for $\frac{n+3}{2} \leq i \leq n$

$g(e_0) = 0$, for $1 \leq i \leq \frac{n+1}{2}$

$g(e_0) = 0$, for $\frac{n+3}{2} \leq i \leq n$

Also the induced face labels are

$g^*(f_1) = 1$
\[ g^*(f_{i+1}) = 0 \quad \text{for} \quad 1 \leq i \leq \frac{n+1}{2} \]
\[ g^*(f_{i+1}) = 1 \quad \text{for} \quad \frac{n+3}{2} \leq i \leq n \]

In view of the above defined labeling pattern, we have
\[ e_i(1) = e_i(0) + 1 = \frac{n(n+2)+1}{2} \quad \text{and} \quad f_i(1) = f_i(0) + 1 = \frac{n+1}{2}. \]

Then \[ |e_i(0) - e_i(1)| \leq 1 \] and \[ |f_i(0) - f_i(1)| \leq 1 \]

Hence G is face integer cordial graph for \( n \geq 3 \).

**Example : 2.7**

The star of cycle \( C_n \) and its face integer cordial labeling of graph is shown in figure 2.7.

![Figure 2.7](http://www.ijmttjournal.org)

**Theorem 2.8**

\( T(P_n) \) is face integer cordial graph for \( n \geq 3 \).

**Proof**:

Let \( v_1, v_2, ..., v_n, u_1, u_2, ..., u_{n-1} \) be vertices, \( e_1, e_2, ..., e_{3n-5} \) be edges and \( f_1, f_2, ..., f_{3n-3} \) interior faces of \( T(P_n) \), where \( e_{2i+1} = v_i u_i, e_{2i+2} = u_i v_{i+1} \) for \( i = 1, 2, ..., n-1 \), \( e_{3n-3i} = u_i u_{i+1} \) for \( i = 1, 2, ..., n-2 \), \( f_i = v_i v_{i+1} v_i \) for \( i = 1, 2, ..., n-1 \) and \( f_{n-1+i} = u_i v_i u_{i+1} u_i \) for \( i = 1, 2, ..., n-2 \).

Let \( G \) be the graph \( T(P_n) \).

Then \[ |V(G)| = 2n-1, |E(G)| = 4n-5 \quad \text{and} \quad |F(G)| = 2n-3. \]

Define \( g : V(G) \rightarrow [-n, ..., n] \) as follows.

**Case (i) :** \( n \) is odd.

\[ g(u_i) = i \quad \text{for} \quad 1 \leq i \leq \frac{n-1}{2} \]
\[ g(u_i) = \frac{n-1}{2} - i \quad \text{for} \quad \frac{n+1}{2} \leq i \leq n \]
\[ g(v_i) = 0 \quad \text{for} \quad 1 \leq i \leq \frac{n-1}{2} \]
\[ g(v_i) = \frac{n-1}{2} + i \quad \text{for} \quad \frac{n+1}{2} \leq i \leq n \]

Also the induced face labels are

\[ g^*(f_{i+1}) = 1 \]
\[ g^*(f_{i+1}) = 0 \quad \text{for} \quad 1 \leq i \leq \frac{n}{2} \]
\[ g^*(f_{i+1}) = 1 \quad \text{for} \quad \frac{n+2}{2} \leq i \leq n \]

In view of the above defined labeling pattern, we have
\[ e_i(0) = e_i(1) = \frac{n(n+2)}{2} \quad \text{and} \quad f_i(1) = f_i(0) + 1 = \frac{n+2}{2}. \]

Then \[ |e_i(0) - e_i(1)| \leq 1 \] and \[ |f_i(0) - f_i(1)| \leq 1 \]

Hence \( G \) is face integer cordial graph for \( n \) is even.
The graph $T(P_n)$ and its face integer cordial labeling is shown in figure 2.8.

**Example 2.8**

In this paper, we prove wheel $W_n$, fan $f_n$, triangular snake $T_n$, double triangular snake $DT_n$, star of cycle $C_n$ and $DS(B_{3n})$ are face integer cordial graph. In the subsequent paper, we will prove vertex switching of cycle, pendant vertex switching of path, helm, closed helm, middle graph of path, total graph of path and subdivision of rim edges of wheel.

**REFERENCES**


[6]. M. Mohamed Sheriff and A. Farhana Abbas, Face Integer Edge Cordial Labeling of Graphs, communicated.


