Odd Graceful Labeling of Some New Type of Graphs

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Abstract

In this paper some new type of odd graceful graphs are investigated. We prove that the graph obtained by joining a cycle $C_8$ with some star graphs $S_{1,r}$ keeping one vertex and three vertices gap between pair of vertices of the cycle admits odd graceful labelling. We also prove that the graph obtained by joining a cycle $C_{12}$ with some star graphs $S_{2,2}$, keeping two, three and five vertices gap between pair of vertices of the cycle admits odd graceful labelling. We observed that the induced function $f^{*}$ assigns each edge a different label. A graph which admits graceful labeling is called a graceful graph.

Definition 1.1 If all the vertices of a graph are assigned some values subject to certain conditions then it is known as graph labelling.

Definition 1.2 A simple graph $G(m, n)$ with $m$ vertices and $n$ edges is graceful if there is a labeling $l$ of its vertices with distinct integers from the set $\{0, 1, 2, \ldots, n\}$ such that the induced edge labeling $e$ defined by $e(uv) = |l(u) - l(v)|$ assigns each edge a different label. A graph which admits graceful labeling is called a graceful graph.

Definition 1.3 A graph $G = (V, E)$ with $q$ edges is said to admit odd graceful labeling if $f: V \rightarrow \{0, 1, 2, \ldots, 2q-1\}$ such that the induced function $f^{*}: E \rightarrow \{1, 3, \ldots, 2q-1\}$ defined as $f^{*}(ab) = |f(a) - f(b)|$ is bijective. A graph which admits odd graceful labeling is called an odd graceful graph.

Definition 1.4 A Star graph is a tree consisting of one vertex adjacent to all others. We denote here a star graph as $S_{1,r}$ where $r$ number of pendant vertices are connected to one vertex.


Main results:

Theorem-1: Graph obtained by joining the cycle $C_8$ with $C_8 \circ S^4_{1,1}$ with one vertex gap four star.
graphs $S_{1,r}$ keeping one vertex gap between pair of vertices of the cycle is odd graceful.

**Proof:** Let $a_1,a_2,a_3,\ldots,a_8$ be the vertices of the cycle $C_8$ and $a'_j$ are pendant vertices of the star graph $S_{1,r}$ adjacent to $a_j$, for $j=1,2,\ldots,r$. We define the vertex labelling as

$$f: V(G) \rightarrow \{0,1,2,\ldots,2q-1\},$$

where $q$ is the total number of edges. As follows

Case-1: If ‘$i$’ is odd, we define $f(x) = \{i-1, \text{if } x = a_i\}$

Let $r$-pendant vertices of the star graph $S_{1,r}$ are connected to each $a_i$ keeping one vertex gap between pair of vertices of the cycle $C_8$.

Case-2: If ‘$i$’ is even, we define $f(a_i) = 2q - i(r + 1) + 1$, if $i < 8$ and $f(a_8) = 1$

$$f(a'_j) = (2q - 1) - (r + 1)(i - 1) - 2(j - 1),$$

except $i = 7, j = r$ for $f(a'_r) = 3$. The vertex function defined induces a bijective edge function $f^*: E(G) \rightarrow \{1,3,5,\ldots,2q-1\}$ Thus $f$ is an edge graceful labeling of $G = C_8 \Theta S_{1,r}^4$. Hence $C_8 \Theta S_{1,r}^4$ is an odd graceful graph. Example:

$C_8 \Theta S_{1,r}^4$ with one vertex gap

**Theorem-2:**

Graph obtained by joining the cycle $C_8$ with three star graphs $S_{1,r}$ keeping three vertices gap between pair of vertices of the cycle is odd graceful.

**Proof:** Let $a_1,a_2,a_3,\ldots,a_8$ be the vertices of the cycle $C_8$. $a'_j$ are pendant vertices of the star graph $S_{1,r}$ adjacent to $a_j$, for $j=1,2,\ldots,r$. We define the vertex labeling as

$$f: V(G) \rightarrow \{0,1,2,\ldots,2q-1\},$$

where $q$ is the total number of edges. As follows

Case-1: If ‘$i$’ is odd, we define $f(x) = \{i-1, \text{if } x = a_i\}$

We define $f(a_i) = 2q - i(r - 2i + 3),i \neq 8$ and $f(a_8) = 1$
The vertex function defined induces a bijective edge function \( f^*: E(G) \rightarrow \{1,3,5,\ldots,2q-1\} \).

Thus \( f \) is an edge graceful labeling of \( G = C_8 \Theta S_{1,r} \). Hence \( C_8 \Theta S_{1,r} \) is an odd graceful graph.

Example:

\[
\begin{align*}
C_8 \Theta S_{1,6} \text{ is odd graceful.}
\end{align*}
\]

**Theorem-4**

\( C_{12} \Theta S_{4,1} \) (where \( r > 4 \)) with one vertex gap between pair of vertices of the cycle \( C_{12} \) is an odd graceful graph.

**Proof:** Let \( a_1, a_2, \ldots, a_{12} \) be the vertices of the cycle \( C_{12} \). \( a_i \) are pendant vertices of the star graph \( S_{1,r} \) adjacent to \( a_i \) for \( j=1,2,\ldots,r \). We define the vertex labelling as

\[
\begin{align*}
f : V(G) & \rightarrow \{0,1,2,\ldots,2q-1\} \text{ where } q \text{ is the total number of edges} \text{ as follows} \\
\text{Case-1: If } 'i' \text{ is odd,} \\
\text{We define } f(a_i) & = i - 1 \\
\text{Case-2: If } 'i' \text{ is even} \\
\text{We define } f(a_i) & = 2q - i(r + 1) + 1 \text{ and}
\end{align*}
\]

The vertex function defined induces a bijective edge function \( f^*: E(G) \rightarrow \{1,3,5,\ldots,2q-1\} \).

Thus \( f \) is a graceful labeling of \( G = C_{12} \Theta S_{4,1} \).

Hence \( C_{12} \Theta S_{4,1} \) is an odd graceful graph.
Example:

\[ C_{12} \odot S_{1,5}^4 \] is odd graceful.

**Theorem 5:** For two vertices gap between pair of vertices of the cycle \( C_{12} \) is an odd graceful graph.

**Proof:** Let \( a_1, a_2, \ldots, a_{12} \) be the vertices of the cycle \( C_{12} \). \( a_i \) are pendant vertices of the star graph \( S_{1,r} \) adjacent to \( a_i \) for \( j=1,2,\ldots,r \). We define the vertex labeling as

\[
f : V(G) \to \{0,1,2,\ldots,2q\text{-}1\}, \text{where } q \text{ is the total number of edges}\]

as follows

**Case 1:** If ‘i’ is odd
We define \( f(a_i) = i - 1 \)

If ‘i’ is even
We define \( f(a_i) = 2q - 2i + 1, i \neq 12 \) and \( f(a_{12}) = 1 \)

**For \( r = 1 \):**
We define

\[
f(a_i^{(j)}) = \begin{cases} 
2q - 2j + 1, & \text{for } i = 1 \\
q + 2j - 4, & \text{for } i = 4 \\
q - 4j + 1, & \text{for } i = 7 \\
q + 2j, & \text{for } i = 10 
\end{cases}
\]

For \( r > 2 \):

\[
f(a_i^{(j)}) = \begin{cases} 
2q - 2j + 1, & \text{for } i = 1 \\
4i + 2j, & \text{for } i = 4 \\
2r + 2j + 9, & \text{for } i = 7 \\
i + 4r + 2j, & \text{for } i = 10 
\end{cases}
\]

The vertex function defined induces a bijective edge function

\[
f^* : E(G) \to \{1,3,5,\ldots,2q-1\}
\]

Thus \( f \) is an edge graceful labeling of \( G = C_{12} \odot S_{1,r}^4 \). Hence \( C_{12} \odot S_{1,r}^4 \) is an odd graceful graph.

Examples:

**Example:**

\[ C_{12} \odot S_{1,5}^4 \] with two vertices gap is odd graceful.
Theorem 6

$C_{12} \otimes S^4_{1,r}$ with two vertices gap is odd graceful

Proof: Let $a_1, a_2, \ldots, a_{10}$ be the vertices of the cycle $C_{12}$ and $a^j_i$ are pendant vertices adjacent to $a_i$, for $j=1, 2, \ldots, r$. We define the vertex labeling as

$$f^*: V(G) \rightarrow \{0, 1, 2, \ldots, 2q-1\},$$

where $q$ is the total number of edges, as follows

- If $i$ is odd, we define $f(a_i) = i - 1$
- If $i$ is even, we define
  $$f(a_i) = \begin{cases} 2q-2r-i+1, & i \leq 4 \\ 2q-4r-i+1, & 4 \leq i \leq 8 \\ 15 & \text{for } i=9 \\
\end{cases}$$
- $f(a_1) = 15$ and $f(a_{12}) = 1$

and

$$f(a^j_i) = \begin{cases} 2q-2j+1, & i = 1 \text{ and } 1 \leq j \leq r \\ 2q-2r-2j-3, & i = 5 \text{ and } 1 \leq j \leq r \\ 2q-4r-2j-7, & i = 9 \text{ and } 1 \leq j < r \\
\end{cases}$$

for $i = 9$ and $j = r$

The vertex function defined induces a bijective edge function $f^*: E(G) \rightarrow \{1, 3, 5, \ldots, 2q-1\}$

Thus $f$ is an edge graceful labeling of $G = C_{12} \cup P_r$. Hence $C_{12} \cup P_r$ is an odd graceful graph.

Theorem 7

$C_{12} \otimes S^4_{1,5}$ with three vertices gap is odd graceful

Proof: Let $a_1, a_2, \ldots, a_{12}$ be the vertices of the cycle $C_{12}$ and $a^j_i$ are pendant vertices adjacent to $a_i$, for $j=1, 2, \ldots, r$. If $q$ is the total number of edges of the graph $G$, we define the vertex labeling as

$$f: V(G) \rightarrow \{0, 1, 2, \ldots, 2q-1\},$$

as follows
We define 
\[ f(a_i) = \begin{cases} 
  i - 1, & \text{if } i \text{ is odd} \\
  2q - i + 1, & \text{if } i \text{ is even and } i < 12 
\end{cases} \]

Case-1: For \( r = 1 \) (only one pendant vertex is attached to each vertex of the cyclic graph \( C_{12} \) keeping five vertices gap in between pair of vertices).

We define 
\[ f(a_{12}) = 3, \quad f(a_1) = 1, \quad f(a_1') = 11 \]

Case-2: For \( r = 2 \) (two pendant vertices are attached to each vertex of the cyclic graph \( C_{12} \) keeping five vertices gap in between pair of vertices).

We define 
\[ f(a_{12}) = 1, \quad f(a_j) = 2j + 1, \text{ for } 1 \leq j \leq 2, \]
\[ f(a_1') = \begin{cases} 
  13 & \text{for } j = 1 \\
  17 & \text{for } j = 2 
\end{cases} \]

Case-3: For \( r > 2 \) (if more than two pendant vertices are attached to each vertex of the cyclic graph \( C_{12} \) keeping five vertices gap in between pair of vertices).

We define 
\[ f(a_{12}) = 1, \quad f(a_j) = \begin{cases} 
  2j + 1, & \text{for } j < 4 \\
  2j + 3, & \text{for } j \geq 4 
\end{cases} \]
\[ f(a_1') = q + 2j - 3, \text{ for } 1 \leq j \leq r \]

The vertex function defined above induces a bijective edge function 
\[ f^*: E(G) \rightarrow \{1, 3, 5, \ldots, 2q - 1\} \]

Thus \( f \) is an edge graceful labeling of \( G = C_{12} \ast S^4_{1,r} \). Hence \( C_{12} \ast S^4_{1,r} \) is an odd edge graceful graph.

\[ C_{12} \ast S^4_{1,r} \text{ with five vertices gap} \]

\[ C_{12} \ast S^4_{1,5} \text{ with five vertices gap is odd graceful} \]

**Theorem-8:** Graph obtained by joining the cycle \( C_{12} \) with a star graph \( S_{1,r} \) is odd graceful.

**Proof:** Let \( a_1, a_2, \ldots, a_8 \) be the vertices of the cycle \( C_{12} \) and \( a_i' \) are pendant vertices of the star graph \( S_{1,r} \) adjacent to \( a_i \) for \( j = 1, 2, \ldots, r \). We define the vertex labeling as
\[ f : V(G) \rightarrow \{0, 1, 2, \ldots, 2q - 1\} \text{ where } q \text{ is the total number of edges} \]

Case-1: If \( i \) is odd
We define 
\[ f(a_i) = i - 1 \]
Let \( r \)-pendant vertices of the star \( S_{1,r} \) are connected to each \( a_i \)

Case-2: If \( i \) is even
We define 
\[ f(a_i) = 2q - i + 1, \quad i \neq 12 \quad \text{and} \quad f(a_{12}) = 1 \]
\[ f(a_j') = 2j + 1, \quad j \neq 4 \text{ and } 1 \leq j \leq r + 1 \]

The vertex function defined induces a bijective edge function 
\[ f^*: E(G) \rightarrow \{1, 3, 5, \ldots, 2q - 1\} \]
Thus \( f \) is an edge graceful labeling of \( G = C_{12} \Theta S_{1,r} \). Hence \( C_{12} \Theta S_{1,r} \) is an odd graceful graph.

Examples:

\[ C_{12} \Theta S_{1,6} \] is odd graceful.

**Application:** Graph labeling has wide application in coding theory, communication networks, optimal circuits layouts and graph decomposition problems.

**References:**