Simple Roots and Weyl Groups

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Abstract — Its origins in the theory of Lie algebras are introduced, and then an axiomatic definition is provided. Simple roots, Bases, Weyl groups, and the transitive action of the latter on the former are explained and proven, respectively.

Keywords — Simple roots, Bases, Weyl groups.

INTRODUCTION


Definition:

Let \( \Phi \) denote a root system of rank \( \ell \) in a Euclidean space \( E \), with Weyl group \( \Delta \). A subset \( \Delta \) of \( \Phi \) is called a base if \( \Delta \) is a linear basis of \( E \) and each root of \( \beta \) can be written as \( \beta = \sum_{\gamma \in \Delta} k_\gamma \gamma \) (with integral coefficients. \( k_\gamma \) all non-negative or all non-positive. The roots in the base \( \Delta \) are called simple roots.

Example:

A root having multiplicity \( n=1 \) is called simple root. \( f(z) = (z - 1)(z - 2) \) has a simple root at \( z_0 = 1 \), but \( g = (z - 1)^2 \) has a root of multiplicity 2 at \( z_0 = 1 \), which is therefore not a simple root.

Lemma:

Let \( \alpha \) be a simple root. Then \( \sigma_\alpha \) permutes the positive roots other than \( \alpha \).

Proof:

Given \( \alpha \) is simple. Let \( \beta \in \Phi \pm \{\alpha\} \).

\[ \beta \neq \alpha \] Both \( \beta \& \alpha \) are positive.

\[ \beta \neq -\alpha \therefore \beta \neq \pm \alpha \]

\[ k_\gamma \neq 0 \text{ for some } \gamma \neq \alpha \]

For, if \( k_\gamma = 0 \forall \gamma \in \Delta \gamma \neq \alpha \) then \( \beta = k_\alpha \alpha \)

\[ \therefore \beta \neq \pm \alpha \] because the only multiple of a root is \( \pm \alpha \).

But \( \beta \neq \pm \alpha \). This is contradiction.

\[ \therefore \text{there exists } \gamma \text{ such that } k_\gamma \neq 0 \]

\[ \sigma_\alpha(\beta) = \beta - (\beta \alpha)\alpha \]

The coefficient of \( \gamma \) in \( \sigma_\alpha(\beta) \) is the same as the coefficient of \( \gamma \) in \( \beta \).

Since \( \sigma_\alpha \) permutes the roots, \( \sigma_\alpha(\beta) \) is also a root.

But the coefficient of \( \gamma \) in \( \sigma_\alpha(\beta) \) is \( k_\gamma \) which is great than 0.

\[ \therefore \text{all the coefficient in } \sigma_\alpha(\beta) \text{ must be positive} \]

\[ \therefore \sigma_\alpha(\beta) \text{ is a positive root.} \]

\[ \sigma_\alpha(-\alpha) = \alpha \& \beta \neq -\alpha \]

\[ \therefore \sigma_\alpha(\beta) \neq \alpha \]

\[ \therefore \sigma_\alpha \text{ permutes the positive roots other than } \alpha. \]

Definition:

A subset \( \Phi \) of all Euclidean space \( E \) is called a root system in \( E \) if the following axioms are satisfied

i) \( \Phi \) spans \( E \), finite, \( 0 \in \Phi \)

ii) If \( \alpha \in \Phi \), the only multiplies of \( \alpha \) in \( \Phi \) are \( \pm \alpha \)

iii) If \( \alpha \in \Phi \), the reflexion \( \sigma_\alpha \) leaves \( \Phi \) invariance. Also if \( \alpha, \beta \in \Phi \) then \( \langle \beta, \alpha \rangle \in \mathbb{Z} \)

Definition:

Let \( \Phi \) be a root system in \( E \). Let \( \mathcal{W} \) denote the subgroup of \( GL(E) \) (group of all invertible endomorphism of \( E \)) generated by the reflexion \( \sigma_\alpha \).

\( \alpha \in \Phi \). Then any \( w \in \mathcal{W} \) is a finite product of reflexion of the form \( \sigma_{\alpha_1}, \sigma_{\alpha_2}, \ldots \sigma_{\alpha_n} \) where \( \alpha_1, \alpha_2, \ldots, \alpha_n \in \Phi \).

\( \alpha \in \Phi \Rightarrow \sigma_{\alpha} \) leaves \( \Phi \) invariance.

\( \therefore w \) leaves \( \Phi \) invariance.

\( \therefore w \) is a permutation of \( \Phi \).

By (i) \( \Phi \) is a finite set, spanning \( E \).

Hence \( \mathcal{W} \) is a subgroup of the symmetric group on \( \Phi \).

Hence \( \mathcal{W} \) is finite.

This \( \mathcal{W} \) is called a weyl group of \( \Phi \).

Note:

\( \mathcal{W} \) is a normal subgroup of \( \text{Aut } \Phi \text{(automorphism of } \Phi) \).

Any element of \( \mathcal{W} \) is a permutation of \( \Phi \).
Lemma:
For all $\sigma \in \mathcal{W}$, $\ell(\sigma) = n(\sigma)$ where $\ell(\sigma)$ is the length of $\sigma$ and $n(\sigma)$ is the number of positive roots $\alpha$ for which $\sigma(\alpha) < 0$.

Proof:
We prove this result using induction on $\ell(\sigma)$.
Suppose $\ell(\sigma) = 0$. Then $\sigma = 1$.

$n(\sigma) = \text{number of positive roots } \alpha \text{ for which } \sigma(\alpha) < 0$

$1(\sigma) = \text{number of positive roots } \alpha \text{ for which } \sigma(\alpha) < 0$

Suppose $\ell(\sigma) = 0$. Then $n(\sigma) = n(\sigma)$ when $\ell(\sigma) = 0$.

Next, we assume that the lemma is true for all $\tau \in \mathcal{W}$ for which $\ell(\tau) = \ell(\sigma)$.

Let $\alpha, \beta, \gamma$ be the reduced expression for $\alpha$.

Then we have $\sigma(\alpha) < 0$.

$n(\sigma) = \text{number of positive roots } \beta \text{ for which } \sigma(\beta) < 0$

$1(\beta) < 0$

Since $\alpha$ is simple, we have $\sigma(\alpha)$ permutes with the roots other than $\alpha$.

$n(\sigma) = n(\sigma) = 1$

$\ell(\sigma) = \ell(\sigma) = 1$

$\ell(\sigma) = \ell(\sigma) = t - 1$

Now $\ell(\sigma) < \ell(\sigma)$.

$\ell(\sigma) = n(\sigma)$.

By induction the result is true for all values of $\ell(\sigma)$.

Lemma:
Let $\Phi$ be a root system in $E$, with weyl group $\mathcal{W}$. If $\sigma \in \text{GL}(E)$, leaves $\Phi$ invariant, then

$\sigma = \sigma \sigma_\alpha \sigma^{-1}$ \hspace{1em} \forall \sigma \in \Phi \land (\beta, \alpha) = (\sigma(\beta), \sigma(\alpha))$ for $\alpha, \beta \in \Phi$.

Proof:
$\sigma$ leaves $\Phi$ invariance.

$\beta \in \Phi \Rightarrow \sigma(\beta) \in \Phi$

$\sigma \in \text{GL}(E) \Rightarrow \sigma$ is invertible.

As $\beta$ runs over $\Phi$, $\sigma(\beta)$ runs over $\Phi$.

Take $\alpha \in \Phi$. By (iii) $\sigma_\alpha$ leaves $\Phi$ invariant.

If $\beta \in \Phi$ then $\sigma_\alpha(\beta) \in \Phi$.

For $\beta \in \Phi, (\sigma \sigma_\alpha \sigma^{-1})(\sigma(\beta)) = \sigma \sigma_\alpha \sigma^{-1}(\sigma(\beta))$

$\sigma = \sigma_\alpha \sigma(\beta) \in \Phi$

$\sigma(\sigma_\alpha(\beta)) = \sigma(\beta) - (\beta, \alpha)\sigma(\alpha)$.

As $\beta$ runs over $\Phi$, $\sigma(\beta)$ runs over $\Phi$.

Consider the hyper plane $\sigma(P_\alpha)$.

Suppose $x \in P_\alpha$. Then $x \in \sigma(\beta)$ for some $\beta \in P_\alpha$.

$\beta \in P_\alpha \Rightarrow (\beta, \alpha) = 0$

$(\sigma \sigma_\alpha \sigma^{-1}(x) = (\sigma \sigma_\alpha \sigma^{-1})(\sigma(\beta))$

$= \sigma \sigma_\alpha \sigma^{-1}(\sigma(\beta))$

$= \sigma \sigma_\alpha(\beta)$

$= \sigma(\sigma_\alpha(\beta))$

$= \sigma(\beta - (\beta, \alpha)\sigma(\alpha)$

$= \sigma(\beta) - (\beta, \alpha)\sigma(\alpha)$

$= \sigma(\beta)$

$= x \land (\beta, \alpha) = 0$

$\sigma \sigma_\alpha \sigma^{-1}(x) = x$

$\sigma \sigma_\alpha \sigma^{-1}$ fixes the hyper plane $\sigma(P_\alpha)$ point wise.

$(\sigma \sigma_\alpha \sigma^{-1}(\sigma(\alpha)) = \sigma \sigma_\alpha(\alpha) = \sigma(-\alpha)$

$\sigma \sigma_\alpha \sigma^{-1}$ leaves $\Phi$ invariant, fixes $\sigma(P_\alpha)$ point wise and stands $\sigma(\alpha)$ to $-\sigma(\alpha)$.

$\sigma \sigma_\alpha \sigma^{-1} = \sigma \sigma(\alpha)$

$\sigma \sigma_\alpha \sigma^{-1}(\sigma(\beta)) = \sigma \sigma(\alpha)$

$= \sigma(\beta) - (\beta, \alpha)\sigma(\alpha)$

$\sigma \sigma_\alpha \sigma^{-1}(\sigma(\beta)) = \sigma \sigma_\alpha(\beta)$

$= \sigma(\beta) - (\beta, \alpha)\sigma(\alpha)$

By (1) & (2)

$(\sigma(\beta), \sigma(\alpha)) = (\beta, \alpha)$

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Bases and Weyl Chambers:

Definition:
Let $\Delta$ be a base of root system $\Phi$. Let $\beta$ be any root. Let $\beta = \sum k_{\alpha} \alpha$. Then the height of $\beta$ relative to $\Delta$ denoted by $ht(\beta)$ is defined by $ht(\beta) = \sum_{\alpha \in \Delta} k_{\alpha} \cdot \alpha$.

$\beta$ is called a positive root if $k_{\alpha} \geq 0$. $\beta$ is called a negative root if $k_{\alpha} \leq 0$.

Lemma:
Let $\alpha, \beta \in E$ we can say that $\beta \preceq \alpha$ if only if either $\alpha - \beta$ is the sum of positive roots or $\alpha = \beta$. This relation $\preceq$ defined above is a partial order on $E$.

Proof:
(i) For all $\alpha \in \Phi$, $\alpha = \alpha$. $\therefore \alpha \preceq \alpha$.
(ii) Let $\alpha \preceq \beta$ and $\beta \preceq \gamma$.

Let $\alpha \preceq \beta$. Then $\sum k_{\alpha} \alpha$ is the sum of positive roots or $\beta = \alpha$.

Let $\beta \preceq \gamma$. Then $\sum k_{\gamma} \gamma$ is the sum of positive roots or $\alpha = \beta$.

Let $\gamma \preceq \delta$. Then $\sum k_{\gamma} \gamma$ is the sum of positive roots or $\beta = \gamma$.

(iii) Let $\alpha \preceq \beta$ and $\beta \preceq \gamma$. Let $\alpha \preceq \beta$. Then $\sum k_{\alpha} \alpha$ is the sum of positive roots or $\beta = \alpha$.

Let $\beta \preceq \gamma$. Then $\sum k_{\beta} \beta$ is the sum of positive roots or $\gamma = \beta$.

Note:
Any positive root is a linear combination of simple roots with nonnegative coefficients. Hence sum of positive roots are also sum of simple roots with nonnegative coefficients. Therefore the above definition of partial order can be replaced by the following definition.

Definition:
For any two elements $\alpha, \beta \in E$ we can say that $\beta \preceq \alpha$ if only if either $\alpha - \beta$ is the sum of simple roots or $\alpha = \beta$.

Lemma II:
Let $\Delta$ be a base of $\Phi$. Let $\alpha, \beta \in \Delta$ be such that $\alpha \preceq \beta$.

Then (i) $(\alpha - \beta) \preceq 0$ and (ii) $(\alpha - \beta)$ is not a root.

Proof:
Suppose $(\alpha - \beta) > 0$. Then $\Delta$ is a base of $\Phi$ and $\alpha, \beta \in \Delta$.

$\therefore \alpha \not\preceq \beta$. $\therefore \alpha \not\preceq \beta$.

Case (i): Suppose $\beta \preceq \alpha$.

Then $(\beta - \alpha)$ is a sum of positive roots or $(\alpha - \beta)$ is a sum of positive roots.

Case (ii): Suppose $\alpha \preceq \beta$.

Then $(\alpha - \beta)$ is a sum of positive roots or $(\beta - \alpha)$ is a sum of positive roots.

$\therefore \alpha \preceq \beta$ and $\beta \preceq \alpha$. $\therefore \alpha \preceq \beta$.

Case (i): Suppose $\beta \preceq \alpha$ and $\gamma \preceq \beta$.

Then $\gamma - \alpha$ is a sum of positive roots.

Case (ii): Suppose $\alpha \preceq \beta$ and $\gamma \preceq \beta$.

Then $\gamma - \alpha$ is a sum of positive roots.

$\therefore \gamma \preceq \alpha$, $\therefore \gamma \preceq \alpha$.

$\therefore \gamma \preceq \alpha$, $\therefore \gamma \preceq \alpha$.

$\therefore \alpha \preceq \beta$, $\therefore \alpha \preceq \beta$.

$\therefore \alpha \preceq \beta$.

$\therefore \alpha \preceq \beta$. $\therefore \alpha \preceq \beta$.

$\therefore \alpha \preceq \beta$. $\therefore \alpha \preceq \beta$.

Step II:
If $\alpha, \beta \in \Delta(\gamma)$ then $\alpha - \beta \preceq 0$ unless $\alpha = \beta$. 

Theorem:
For each vector $\gamma \in E$, we define $\Phi^+(\gamma) = \{\alpha \in \Phi : \langle \alpha, \gamma \rangle > 0\}$. The set of all roots lying in the positive side of the hyper plane orthogonal to $\gamma$.

Proof:
Let $\gamma$ be regular and $\Delta(\gamma)$ be the set of all indecomposable roots in $\Phi^+(\gamma)$ be a base of $\Phi$ and every base is obtained in this way.

Step I:
We claim that each root in $\Phi^+(\gamma)$ is a non negative $\preceq$-linear combination of elements of $\Delta(\gamma)$.

Suppose not

Then there exists an $\alpha \in \Phi^+(\gamma)$, which cannot be written as a non negative $\preceq$-linear combination of elements of $\Delta(\gamma)$.

Choose an $\alpha$ such that $\langle \gamma, \alpha \rangle$ is as small as possible. Suppose $a \in \Delta(\gamma)$. Then $\alpha \preceq a$. $\alpha$ is a non negative $\preceq$-linear combination of elements of $\Delta(\gamma)$.

This is a contradiction.

$\therefore \alpha \in \Delta(\gamma)$.

$\therefore \alpha$ is decomposable.

Let $\alpha = \beta_1 + \beta_2$ where $\beta_1, \beta_2 \in \Phi^+(\gamma)$.

$\langle \gamma, \alpha \rangle = \langle \gamma, \beta_1 + \beta_2 \rangle = \langle \gamma, \beta_1 \rangle + \langle \gamma, \beta_2 \rangle > 0$

Also $\langle \gamma, \beta_1 \rangle < \langle \gamma, \alpha \rangle$ and $\langle \gamma, \beta_2 \rangle < \langle \gamma, \alpha \rangle$.

$\therefore \beta_1 + \beta_2$ is a non-negative $\preceq$-linear combination of elements of $\Delta(\gamma)$.

This is a contradiction.

$\therefore$ each root of $\Phi^+(\gamma)$ is a non-negative $\preceq$-linear combination of elements of $\Delta(\gamma)$.

Hence our claim.

Step II:
If $\alpha, \beta \in \Delta(\gamma)$ then $\alpha - \beta \preceq 0$ unless $\alpha = \beta$. 


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Suppose $\alpha, \beta \in \Delta(y)$, $\alpha \neq \beta$.

Suppose $(\alpha, \beta) > 0$. Then $\alpha, \beta$ be non-proportional roots. If $(\alpha, \beta) > 0$ (i.e., if the angle between $\alpha, \beta$ is acute) then $\alpha - \beta$ is a root. If $(\alpha, \beta) > 0$ then $\alpha + \beta$ is a root.) $\alpha - \beta$ is a root.

Suppose $\beta = -\alpha$.
Then $(\alpha, \beta) = (\alpha, -\alpha) = (\alpha, \alpha) \leq 0$.
Which is contradiction to $(\alpha, \beta) > 0$.
\[ \therefore \beta = 0 \]
\[ \therefore \text{ either } \beta - \alpha \text{ or } \alpha - \beta \text{ is in } \Phi^+(y). \]

case (i) Suppose $\alpha - \beta \in \Phi^+(y)$.
Then $\alpha = \beta + (\alpha - \beta)$.
$\therefore \alpha$ is decomposable.
$\therefore \alpha \in \Phi^+(y)$. This is contradiction.

case (ii) Suppose $\beta - \alpha \in \Phi^+(y)$.
Then $\beta = \beta + (\alpha - \beta)$.
$\therefore \beta$ is decomposable. $\therefore \beta \in \Phi^+(y)$.
This is contradiction.
$\therefore \alpha - \beta \leq 0$
Thus if $\alpha, \beta \in \Delta(y)$ then $\alpha - \beta \leq 0$ unless $\alpha = \beta$.

Step III:
$\Delta(y)$ is linearly independent.

Suppose $\sum r_\alpha \alpha = 0$ where $\alpha \in \Delta(y)$ & $r_\alpha \in \mathbb{R}$.
Separate the indices $\alpha$ for which $r_\alpha > 0$ and those for which $r_\alpha < 0$.
Then we can write this as $\sum r_\alpha \alpha = \sum t_\beta \beta$ where $r_\alpha > 0$ & $t_\beta = 0$.
The set of $\alpha$’s and $\beta$’s are distinct.
$\therefore \alpha \neq \beta \forall \alpha, \beta$

Let $\varepsilon = \sum r_\alpha \alpha$
$\langle \varepsilon, \varepsilon \rangle = \sum r_\alpha \alpha \cdot \sum r_\alpha \alpha$

$= \sum r_\alpha \alpha \cdot \sum t_\beta \beta$ $= \sum r_\alpha t_\beta (\alpha, \beta) \leq 0$ (by step II)

$\therefore 0 \leq \langle \varepsilon, \varepsilon \rangle \leq 0$

$\therefore \langle \varepsilon, \varepsilon \rangle = 0$ $\Rightarrow \varepsilon = 0$.

$0 = \langle \varepsilon, \varepsilon \rangle = \langle \varepsilon, \sum r_\alpha \alpha \rangle$
$= \sum r_\alpha \langle \varepsilon, \alpha \rangle$ \hspace{1cm} (3)
$\sum r_\alpha \langle \varepsilon, \alpha \rangle = 0$
$\alpha \in \Delta(y) \Rightarrow \langle \varepsilon, \alpha \rangle > 0 \forall \alpha$

Also $t_\beta > 0$ for each $\alpha$.
$\therefore \sum r_\alpha \langle \varepsilon, \alpha \rangle > 0$ \hspace{1cm} (4)

By (3) and (4) we get $0 > 0$. This is contradiction.
Similarly $t_\beta = 0$ $\forall \beta$.
$\therefore \Delta(y)$ is linearly independent.

Step IV:
$\Delta(y)$ is a base of $\Phi$
Now $y$ is regular.
$\therefore \Phi = \Phi^+(y) \cup -\left(\Phi^+(y)\right)$

$\beta \in \Phi$. Then $\beta = 0$ & $\beta \in \Phi^+(y)$ or $\beta \in \Phi^+(y)$ but not both.

Suppose $\beta \in \Phi^+(y)$
Then by step I, $\beta = \sum k_\alpha \alpha$ where $\alpha \in \Delta(y)$.
$k_\alpha \in \mathbb{R}$, $k_\alpha > 0$.

Suppose $\beta \in -\Phi^+(y)$
Then $-\beta \in \Phi^+(y)$ and so by step I, $-\beta = \sum t_\alpha \alpha$ where $\alpha \in \Delta(y)$.
t_\alpha \in \mathbb{R}$, $t_\alpha > 0$.
$\therefore \beta = \sum t_\alpha \alpha$ where $t_\alpha \in \mathbb{R}$, $t_\alpha > 0$.
Thus in either case, we have $\beta = \sum k_\alpha \alpha$ where $\alpha \in \Delta(y)$ $k_\alpha \in \mathbb{R}$ such that $k_\alpha$ are all non-negative or non-positive.
Each $\beta \in \Phi^+(y)$ is a linear combination of elements of $\Delta(y)$.
$\therefore \Delta(y)$ spans $\Phi^+(y)$
$\Phi$ spans $E$.
$\therefore$ every element of $E$ is a linear combination of elements of $\Phi$.

Let $x \in E$. Then $\sum a_\alpha \alpha$

$\therefore \sum a_\alpha \alpha = \sum t_\beta \beta$

$\sum a_\alpha \alpha = \sum t_\beta \beta$ $= \sum a_\alpha + \sum t_\beta (-\beta)$

$\therefore a$ is a linear combination of elements of $\Phi^+(y)$
$\therefore \Phi^+(y)$ spans $E$
$\therefore \Delta(y)$ spans $\Phi^+(y)$
$\therefore \Delta(y)$ spans $E$
$\therefore \Delta(y)$ is a linear combination of elements of $E$
$\therefore \Delta(y)$ is a base of $\Phi$
So, $\Phi$ has a basis $\Delta(y)$

Step V:
Each base $\Delta$ of $\Phi$ is of the form $\Delta(y)$
Let $\Delta = \{y_1, y_2, \ldots, y_n\}$ be a base of $\Phi$
Then $\delta_i$ be the projection of $y_i$ on the sub space $E_i$ spanned by all basis vectors except $y_i$
$\delta_i = \{y_1, y_2, \ldots, y_{i-1}, y_i, \ldots, y_n\}$
Each $E_i$ is a hyper plane of $E$ containing $\delta_i$.
Let $y = \sum n_i \delta_i n_i > 0$
$\langle x, y \rangle = \langle \sum n_i \delta_i, n_i \rangle$
$= \sum n_i (\delta_i, n_i) > 0$
Hence $\exists y \in E$ such that $\langle y, y \rangle > 0 \forall y$.

Let $\beta \in \Phi$. Then $\beta = \sum k_i y_i \in \Delta$ & $k_i \neq 0$
\[ (\gamma, \beta) = \left( \sum_{i=1}^{n} k_i \gamma_i, \sum_{i=1}^{n} k_i \gamma_i \right) \]

\[ = \sum_{i=1}^{n} k_i (\gamma, \gamma_i) > 0 \text{ if the } k_i \text{'s are positive} \]

\[ < 0 \text{ if the } k_i \text{'s are negative} \]

\[ \therefore (\gamma, \beta) \neq 0. \quad \therefore \gamma \notin \mathbb{P}_g \therefore \beta \notin \Phi \]

\[ \therefore \gamma \text{ is regular.} \]

Let \( \beta \in \Phi^\ast \). Then \( \beta \) is a positive root.

Let \( \beta = \sum_{i=1}^{n} k_i \gamma_i \) where \( k_i \in \mathbb{R}, k_i > 0 \)

\[ \forall i = 1, 2, \ldots, n \]

\[ (\gamma, \beta) = (\sum_{i=1}^{n} k_i \gamma_i, \sum_{i=1}^{n} k_i \gamma_i) > 0 \]

\[ \therefore \beta \in \Phi^\ast (\gamma) \]

\[ \therefore \Phi^\ast \subset \Phi^\ast (\gamma) \]

Let \( \beta \in \Phi^\ast (\gamma) \). Then \( (\gamma, \beta) > 0 \).

Let \( \beta = \sum_{i=1}^{n} k_i \gamma_i \) where \( k_i \)'s are all non-negative or non-positive.

\[ \therefore (\gamma, \sum_{i=1}^{n} k_i \gamma_i) > 0 \]

Suppose all the \( k_i \)'s are non-positive.

Then \( \beta \in \Phi^\ast \subset -\Phi^\ast \subset -\Phi^\ast (\gamma) \)

\[ \therefore -\Phi^\ast (\gamma) \text{. This is contradiction for } \beta \in \Phi^\ast (\gamma) \]

\[ \therefore \text{all the } k_i \text{'s are non-negative.} \]

\[ \therefore \beta \in \Phi^\ast \]

\[ \beta \in \Phi^\ast (\gamma) \Rightarrow \beta \in \Phi^\ast \]

\[ \therefore \Phi^\ast (\gamma) \subset \Phi^\ast \]

Hence \( \Phi^\ast = \Phi^\ast (\gamma) \)

Let \( \alpha \in \Delta \). Then \( \alpha = \gamma_i \) for some i.

\[ = 0, \gamma_1 + 0, \gamma_2 + \ldots + 1, \gamma_i + \ldots + 0, \gamma_n \]

\[ \therefore \alpha \in \Phi^\ast \text{ for the coefficient are all non-negative.} \]

\[ \therefore \alpha \in \Delta \Rightarrow \alpha \in \Phi^\ast = \Phi^\ast (\gamma) \]

\[ \therefore \Delta \subset \Phi^\ast (\gamma) \]

Let us suppose that \( \alpha \in \Delta \) & \( \alpha \) is decomposable.

Then \( \alpha = a_1 + a_2 \)

\[ \therefore a_1, a_2 \in \Phi^\ast (\gamma) \]

Let \( a_1 = \sum_{i=1}^{n} t_i \gamma_i \), \( a_2 = \sum_{i=1}^{n} s_i \gamma_i \)

Then \( \alpha = \sum_{i=1}^{n} (t_i + s_i) \gamma_i \), \( (t_i + s_i) > 0 \)

\[ \therefore \text{any element } \alpha \text{ of } \Delta \text{ is a linear combination of other elements of } \Delta \text{ which is a contradiction for } \Delta \text{ is a linearly independent.} \]

\[ \therefore \alpha \text{ is indecomposable} \]

\[ \alpha \in \Delta (\gamma) \]

\[ \alpha \in \Delta = \alpha \in \Delta (\gamma) \Rightarrow \Delta \subset \Delta (\gamma) \]

But \( \text{Card } \Delta = n = \text{Card } \Delta (\gamma) \)

\[ \therefore \Delta = \Delta (\gamma) \]

**REFERENCES**


