Forcing Edge Detour Domination Number of Graphs

Mahalakshmi A, Palani K, Somasundaram S

Department of Mathematics, Sri Sarada College for Women, Tirunelveli 627 011, Tamil Nadu, India.
Department of Mathematics, A.P.C Mahalakshmi College For Women, Tuticorin 628 002, Tamil Nadu, India.
Department of Mathematics, Manonmaniam Sundaranar University, Tirunelveli 627 012, Tamil Nadu, India.

Abstract. In this paper, we introduce the new concept, forcing edge detour domination number and obtain the forcing edge detour domination number for some well known graphs.

Keywords — Domination, forcing edge detour and forcing edge detour domination number.

I. INTRODUCTION

The concept of domination was introduced by Ore and Berge [7]. Let G be a finite, undirected connected graph with neither loops nor multiple edges. A subset D of V(G) is a dominating set of G if every vertex in V-D is adjacent to at least one vertex in D. The minimum cardinality among all dominating sets of G is called the domination number \( \gamma(G) \) of G. For basic definitions and terminologies, we refer Harary [1]. (G, D)-number of a graph was introduced by Palani.K and Nagarajan.A [8]. Let G = (V, E) be any connected graph with at least two vertices. A subset S of V (G) which is both dominating and geodetic set of G is called a (G, D)-set of G. For vertices u and v in a connected graph G, the distance D(u,v) is the length of longest u-v path in G. A u-v path of length D(u,v) is called a u-v detour.

A subset S of V is called a detour set if every vertex in G lies on a detour joining a pair of vertices of S. The detour number dn(G) of G is the minimum order of a detour set and any detour set of order dn(G) is called a detour basis of G. These concepts were studied by Chartrand [3]. A subset S of V is called an edge detour set of G if every edge in G lies on a detour joining a pair of vertices of S. The edge detour number edn(G) of G is the minimum order of an edge detour set and any edge detour set of order edn(G) is an edge detour basis of G. A graph G is called an edge detour graph if it has an edge detour set. Edge detour graphs were introduced and studied by Santhakumaran and Athisayanathan [10]. Forcing (G, D)-number of a graph was introduced and studied by Palani.K and Nagarajan.A [9]. Let G be a connected graph and S be a \( \gamma(G) \)-set of G. A subset T of S is called a forcing subset for S if S is the unique \( \gamma(G) \)-set of G containing T. A forcing subset T of S with minimum cardinality is called a minimum forcing subset for S. The forcing (G, D)-number of S, denoted by \( f_{GD}(S) \), is the cardinality of a minimum forcing subset of S. The forcing(G,D)-number of G is the minimum of \( f_{GD}(S) \), where the minimum is taken is over all \( \gamma(G) \)-sets S of G and it is denoted by \( f_{GD}(G) \). That is, \( f_{GD}(G) = \min \{|f_{GD}(S) : S \text{ is any } \gamma(G) \text{-set of } G| \}

An edge detour dominating set is a subset S of V (G) which is both a dominating and an edge detour set of G. An edge detour dominating set is said to be a minimal edge detour dominating set of G if no proper subset of S is an edge detour dominating set of G. An edge detour dominating set S is said to be minimum edge detour dominating set of G if there exists no edge detour dominating set S’ such that \( |S| < |S'| \). The smallest cardinality of an edge detour dominating set of G is called the edge detour domination number of G. It is denoted by \( \gamma_{e,D}(G) \).

Any edge detour dominating set S of G of minimum cardinality \( \gamma_{e,D}(G) \) is called an edge detour dominating set S of G. An edge detour dominating set of G is called a forcing edge detour dominating set if there exists no edge detour dominating set S’ of G such that \( |S| < |S'| \). The smallest cardinality of a forcing edge detour dominating set of G is called the forcing edge detour dominating number of G. It is denoted by \( \gamma_{e,D}^*(G) \).

Theorem 1. \( \gamma_{e,D}(K_{p,n}) = n \).

Theorem 2. \( \gamma_{e,D}(P_n) = \begin{cases} \frac{n-4}{3} + 2 & \text{if } n \geq 5 \\ 2 & \text{if } n = 2, 3 \text{ or } 4. \end{cases} \)

Theorem 3. \( \gamma_{e,D}(C_n) = \begin{cases} \left\lfloor \frac{n}{3} \right\rfloor & \text{if } n > 5 \\ 2 & \text{if } n = 2, 3 \text{ or } 4. \end{cases} \)

Remark 1. If the set of all pendant vertices of a graph G forms an edge detour dominating G, then S is the unique minimum edge detour dominating set of G.
II. FORCING EDGE DETOUR DOMINATION NUMBER OF A GRAPH

Definition 2.1. Let G be an edge detour dominating graph and S be an edge detour dominating basis of G. A subset T ⊆ S is called a forcing (γ, eD) subset for S if S is the unique edge detour dominating basis containing T. The forcing (γ, eD) - number of S denoted by fγ,eD(S), is the cardinality of a minimum forcing subset for S. The forcing edge detour domination number of G is denoted by fγ,eD(G) is fγ,eD(G) = min { fγ,eD(S) }, where the minimum is taken over all edge detour dominating bases S in G.

Observation 2.2. For every edge detour dominating graph, 0 ≤ fγ,eD(G) ≤ γ,eD(G).

Example 2.3. i) For a graph 2.1 (a), {v1, v4} is a unique edge detour dominating basis and so γ,eD(G) = 2 = dn1(G). Therefore, fγ,eD(G) = 0.

ii) For the graph G in the given figure 2.1 (b), S1 = {u, x, y}, S2 = {u, x, z} are the edge detour dominating bases of G {x} and {z} are the forcing subsets of S1 and S2 respectively. Hence, fγ,eD(G) = 1.

iii) For the graph G in Figure 2.1(c), S1 = {u, y, z}, S2 = {u, v, x}, S3 = {v, x, z}, S4 = {x, y, z}, S5 = {w, x, z} are the six edge detour dominating bases of G. And every single element appears in atleast two of the edge detour dominating sets. And also, {u, v}, {u, y}, {u, w}, {u, x}, {u, z}, {v, w}, {v, z}, contained in only one of the six edge detour dominating bases. Therefore, fγ,eD(G) = 2.

Proposition 2.10. If S is a forcing edge detour dominating basis, then fγ,eD(S) = 0 if n ≡ 1 (mod 3).
Let \( P_n = \{v_1, v_2, \ldots, v_{3k+1}\} \), \( k > 0 \), \( S = \{v_1, v_4, \ldots, v_{3k+1}\} \) is the unique edge detour dominating set of \( P_n \). Therefore, by observation 2.5(1), \( f_{eD}^r(P_n) = 0 \).

**Proposition 2.11.** Every three element subsets of \( V(K_p) \) are the edge detour dominating bases of \( K_p \), for \( p \geq 3 \).

**Proof.** Let \( S = \{u, v, w\} \) be a three element subset of \( V(K_p) \). Every edge other than uv lie in some edge detour joining u and v. And uv is in some edge detour joining v and w. Also, S dominates all the vertices of \( K_p \). Therefore, S is an edge detour dominating set of \( K_p \).

**Claim:** No two element subset of \( V(K_p) \) is an edge detour dominating set of \( K_p \). Suppose, let \( S' = \{u', v'\} \) be an edge detour dominating set of \( K_p \). Clearly, the edge u've lie in no edge detour joining u' and v'. Therefore, no two element subset of \( V(K_p) \) is an edge detour dominating set of \( K_p \). Hence, S is an edge detour dominating basis of \( K_p \).

**Theorem 2.12.** Let \( G \) be a complete graph \( K_p \) of order \( p \geq 4 \). Then, \( \gamma_{eD}(K_p) = 3 \) and \( f_{eD}^r(K_p) = 3 \).

**Proof.** Let \( G \) be a complete graph \( K_p \). Let \( p \geq 4 \). By Proposition 2.11, \( \gamma_{eD}(K_p) = 3 \). Since every three element subset of \( V(K_p) \) is an edge detour dominating basis and no edge detour dominating basis of G is the unique edge detour dominating basis containing any of its proper subsets. Therefore, by observation 2.5(3), \( f_{eD}^r(K_p) = 3 \).

**Remark 2.13.** When \( p = 3 \), \( \{u, v, w\} \) is a unique edge detour dominating basis. Therefore, \( \gamma_{eD}(K_3) = 3 \). And so by observation 2.5(1), \( f_{eD}^r(K_3) = 0 \).

**Theorem 2.14.** Let \( G \) be a cycle \( C_n \) of order \( n = 3k \), \( k > 1 \). Then, \( \gamma_{eD}(C_n) = 1 \).

**Proof.** Let \( n = 3k \), \( k \geq 1 \). Let \( V(C_n) = \{v_1, v_2, \ldots, v_{3k}\} \). Clearly, \( S_1 = \{v_1, v_4, \ldots, v_{3(k-1)+1}\} \), \( S_2 = \{v_2, v_5, \ldots, v_{3k+1}\} \) and \( S_3 = \{v_3, v_6, \ldots, v_{3k}\} \) are the only three edge detour dominating bases of \( V(C_n) \). Clearly, for \( i = 1 \) to \( n \), \( \{v_i\} \subseteq S_i \) for exactly one \( j \) such that \( 1 \leq j \leq 3 \). Hence, \( f_{eD}^r(C_n) = 1 \).

**Corollary 2.15.** Let \( G \cong K_{m,n} \) (\( 2 \leq m \leq n \)). Suppose \( S \subseteq V \) is an edge detour dominating basis of \( G \). Then, if \( m = 1 \) and \( n > 1 \) (or \( n = 1 \) and \( m > 1 \)) then, \( \gamma_{eD}(K_{m,n}) = n \) (or \( m \)) and \( f_{eD}^r(K_{m,n}) = 0 \).

**Proof.** Let \( G = K_{m,n} \) with bipartition \( V_1 = \{a_1, a_2, \ldots, a_m\} \) and \( V_2 = \{b_1, b_2, \ldots, b_n\} \).

**i)** If \( m = 1 \) and \( n > 1 \) (or \( n = 1 \) and \( m > 1 \)) then, \( \gamma_{eD}(K_{m,n}) = n \) (or \( m \)) and \( f_{eD}^r(K_{m,n}) = 0 \).

**Proof.** Let \( G = K_{m,n} \) with bipartition \( V_1 = \{a_1, a_2, \ldots, a_m\} \) and \( V_2 = \{b_1, b_2, \ldots, b_n\} \).

**ii)** If \( m = 1 \) and \( n = 1 \) then, \( \gamma_{eD}(K_{m,n}) = 1 \) and \( f_{eD}^r(K_{m,n}) = 0 \).

**Corollary 2.16.** Let \( T \) be a tree. If the set of all end vertices of \( T \) forms an edge detour dominating set of \( G \) then, \( f_{eD}^r(T) = 0 \).

**Proof.** By the remark 1.5, the set of all end vertices of a tree is the unique edge detour dominating basis of \( T \). Therefore, \( f_{eD}^r(T) = 0 \).

**Theorem 2.17.** For each pair \( a, b \) of integers with \( 0 \leq a \leq b \), and \( a \) is even, there is an edge detour dominating graph with \( f_{eD}^r(G) = a \) and \( \gamma_{eD}(G) = b \).

**Proof.** Let \( G \) be the graph \( P_s \). Attach \( r \) and \( s \) end vertices to \( P_s \) such that \( r + s = b \). Then, the set of all end vertices of \( P_s \) is the unique edge detour dominating set. Therefore, \( \gamma_{eD}(G) = b \). By Observation 2.5(1), \( f_{eD}^r(G) = 0 \).

**Case 2:** \( a \geq 1 \). Consider \( H = K_{1,b-a} \) be the star with end vertices \( t_1, t_2, \ldots, t_{b-a} \). Suppose for \( i \) to be 1 to \( a/2 \), \( F_i = C_3(x_i, y_i, z_i, u_i, v_i, x_i) \) be the cycle. Let \( v \) be the central vertex of \( H \). By an edge, attach \( a/2 \) cycles \( F_i \) of length 5 to \( v \). The graph is as in Figure 2.2.
Figure 2.2

Claim 1: \( \gamma_{ed}(G) = b \). Obviously, \( \{t_1, t_2, \ldots, t_{a/2}, u_x, x_i\} \) is an edge detour dominating set of \( G \). Therefore, \( \gamma_{ed}(G) \leq b - a + 2(a/2) = b \). 
---------------------

To dominate the vertices of each of the \( a/2 \) cycles, we need at least two vertices from each cycle. Also, \( \{t_1, t_2, \ldots, t_{a/2}\} \subseteq S \) for all \( \gamma_{ed} \)-set of \( G \). Therefore, \( \gamma_{ed}(G) \leq b - a + 2(a/2) = b \). 
---------------------

By (1) and (2), \( \gamma_{ed}(G) = b \).

Claim 2: \( f_{\gamma_{ed}}(G) = a \).

Since, \( \{t_1, t_2, \ldots, t_{a/2}\} \) is the set of all edge detour dominating vertices, by Theorem 2.7, \( f_{\gamma_{ed}}(G) \leq b - (b - a) \). Clearly, a set \( S \) is an edge detour dominating basis of \( G \) if and only if \( S = \{t_1, t_2, \ldots, t_{a/2}\} \cup \{v_i, s_i\mid 1 \leq i \leq a/2\} \) where \( \{v_i, s_i\} \) is a dominating set of \( F_i \). It is obvious that any set \( T \) which is a proper subset of \( S = \{t_1, t_2, \ldots, t_{a/2}\} \) is contained in at least two edge detour dominating basis and \( \{x_1, u_1, x_2, u_2, \ldots, x_{a/2}, u_{a/2}\} \) is a subset of \( S = \{t_1, t_2, \ldots, t_{a/2}\} \) such that \( S \) is edge detour dominating basis containing it. Therefore, 
\[ f_{\gamma_{ed}}(G) = \left| x_1, u_1, x_2, u_2, \ldots, x_{a/2}, u_{a/2}\right| = 2(a/2) = a. \]

REFERENCES