Some New Families of Sum Divisor Cordial Graphs

P. Lawrence Rozario Raj*1 and S. Hema Surya*2

*P.G. and Research Department of Mathematics, St. Joseph’s College, Tiruchirappalli – 620 002, Tamil Nadu, India.

M.Phil Scholar, P.G. and Research Department of Mathematics, St. Joseph’s College, Tiruchirappalli – 620 002, Tamil Nadu, India.

Abstract - A sum divisor cordial labeling of a graph G with vertex set V is a bijection f from V(G) to \{1, 2, ..., |V(G)|\} such that an edge uv is assigned the label 1 if 2 divides f(u)+f(v) and the label 0 otherwise, then the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. A graph with a sum divisor cordial labeling is called a sum divisor cordial graph. In this paper, we investigate some new families of sum divisor cordial graphs. We will provide brief results of the sum divisor cordial labeling of switching of a vertex labeling G. We begin with simple, finite, undirected graph G = (V(G),E(G)) with p vertices and q edges. For standard terminology and notations related to graph theory we refer to Chartrand [2] while for number theory we refer to Burton [3]. Graph labelings have enormous applications within mathematics as well as to several areas of computer science and communication networks. For a dynamic survey on various graph labeling problems along with an extensive bibliography we refer to Gallian [4]. The concept of cordial labeling was introduced by Cahit [1]. Varatharajan et al.[6] introduced the concept of divisor cordial labeling of graphs. The concept of sum divisor cordial labeling was introduced by Lourdusamy et al.[5]. The present work is focused on some new families of sum divisor cordial graphs. We will provide brief summary of definitions and other information which are necessary for the present investigations.

Keywords - Divisor cordial labeling, sum divisor cordial labeling, sum divisor cordial graph.

I. INTRODUCTION

We begin with simple, finite, undirected graph G = (V(G),E(G)) with p vertices and q edges. For standard terminology and notations related to graph theory we refer to Chartrand [2] while for number theory we refer to Burton [3]. Graph labelings have enormous applications within mathematics as well as to several areas of computer science and communication networks. For a dynamic survey on various graph labeling problems along with an extensive bibliography we refer to Gallian [4]. The concept of cordial labeling was introduced by Cahit [1]. Varatharajan et al.[6] introduced the concept of divisor cordial labeling of graphs. The concept of sum divisor cordial labeling was introduced by Lourdusamy et al.[5]. The present work is focused on some new families of sum divisor cordial graphs. We will provide brief summary of definitions and other information which are necessary for the present investigations.

Definition :1.1

A graph labeling is the assignment of unique identifiers to the edges and vertices of a graph.

Definition :1.2

A mapping f : V(G) → {0,1} is called binary vertex labeling of G and f(v) is called the label of the vertex v of G under f. If for an edge e = uv, the induced edge labeling f* : E(G) → \{0,1\} is given by f*(e) = |f(u) − f(v)|. Then \( \nu_0(i) \) = number of vertices of having label i under f and \( \nu_1(i) \) = number of edges of having label i under f*.

Definition :1.3

A binary vertex labeling f of a graph G is called a cordial labeling if \( |\nu_0(0) − \nu_0(1)| \leq 1 \) and \( |\nu_1(0) − \nu_1(1)| \leq 1 \). A graph G is cordial if it admits cordial labeling.

Definition :1.4

Let a and b be two integers. If a divides b means that there is a positive integer k such that b = ka. It is denoted by a \mid b. If a does not divide b, then we denote a \nmid b.

Definition :1.5

Let \( G = (V(G), E(G)) \) be a simple graph and f : V(G)→\{1,2,...,|V(G)|\} be a bijection. For each edge uv, assign the label 1 if f(u)+f(v) and the label 0 otherwise. The function f is called a divisor cordial labeling if \( |\nu_0(0) − \nu_0(1)| \leq 1 \). A graph with a divisor cordial labeling is called a divisor cordial graph.

Definition :1.6

Let \( G = (V(G), E(G)) \) be a simple graph and f : V(G)→\{1,2,...,|V(G)|\} be a bijection. For each edge uv, assign the label 1 if 2|(f(u)+f(v)) and the label 0 otherwise. The function f is called a sum divisor cordial labeling if \( |\nu_0(0) − \nu_0(1)| \leq 1 \). A graph which admits a sum divisor cordial labeling is called a sum divisor cordial graph.

Definition :1.7

A vertex switching \( G_v \) of a graph G is obtained by taking a vertex v of G, removing the entire edges incident with v and adding edges joining v to every vertex which are not adjacent to v in G.

Definition :1.8

Let \( G = (V(G), E(G)) \) be a graph with vertex set \( V = S_1 \cup S_2 \cup \ldots \cup S_t \cup T \) where each \( S_i \) is a set of vertices having at least two vertices of the same degree and \( T = V \setminus S_i \). The degree splitting graph of G denoted by DS(G) is obtained from G by adding vertices \( w_1, w_2, w_3, \ldots, w_t \) and joining to each vertex of \( S_i \) for 1 ≤ i ≤ t.

II. MAIN THEOREMS

Theorem : 2.1

Given a positive integer n, there is a sum divisor cordial graph G which has n vertices.
Proof.
Case 1: Suppose \( n \) is even.

Construct a graph \( G \) of \( n \) vertices \( v_1, v_2, \ldots, v_n \) in the following manner. Form a path containing \( \frac{n}{2} \) vertices \( v_1, v_2, \ldots, v_{\frac{n}{2}} \), then attach \( \frac{n}{2} \) vertices \( v_{\frac{n}{2}+1}, \ldots, v_n \) to the vertex \( v_1 \).

Define \( f: V(G) \to \{1,2,\ldots,n\} \) as follows
\[
\begin{align*}
  f(v_i) &= 2i - 1 & \text{for } & 1 \leq i \leq \frac{n}{2}, \\
  f\left(v_{\frac{n}{2}+i}\right) &= 2i & \text{for } & 1 \leq i \leq \frac{n}{2}.
\end{align*}
\]

Then, the induced edge labeling \( f^* \) is as follows.
\[
\begin{align*}
  f^*(v_ivi_{i+1}) &= 1 & \text{for } & 1 \leq i \leq \frac{n}{2} - 1, \\
  f^*(v_1v_i) &= 0 & \text{for } & \frac{n}{2} + 1 \leq i \leq n.
\end{align*}
\]

Thus, \( e_i(0) = \frac{n}{2} \) and \( e_i(1) = \frac{n}{2} - 1 \).

Hence \( |e_i(0) - e_i(1)| \leq 1 \).

Therefore, the resultant graph \( G \) is sum divisor cordial for \( n \) is even.

Case 2: Suppose \( n \) is odd.

Construct a graph \( G \) of \( n \) vertices \( v_1, v_2, \ldots, v_n \) in the following manner. Form a path containing \( \frac{n+1}{2} \) vertices \( v_1, v_2, \ldots, v_{\frac{n+1}{2}} \), and then attach \( \frac{n-1}{2} \) vertices \( v_{\frac{n+1}{2}+1}, \ldots, v_n \) to the vertex \( v_1 \).

Define \( f: V(G) \to \{1,2,\ldots,n\} \) as follows
\[
\begin{align*}
  f(v_i) &= 2i - 1 & \text{for } & 1 \leq i \leq \frac{n+1}{2}, \\
  f\left(v_{\frac{n+1}{2}+i}\right) &= 2i & \text{for } & 1 \leq i \leq \frac{n-1}{2}.
\end{align*}
\]

Then, the induced edge labeling \( f^* \) is as follows.
\[
\begin{align*}
  f^*(v_ivi_{i+1}) &= 1 & \text{for } & 1 \leq i \leq \frac{n-1}{2}, \\
  f^*(v_1v_i) &= 0 & \text{for } & \frac{n+3}{2} \leq i \leq n.
\end{align*}
\]

Thus, \( e_i(0) = \frac{n-1}{2} \) and \( e_i(1) = \frac{n-1}{2} \).

Hence \( |e_i(0) - e_i(1)| \leq 1 \). Therefore, the resultant graph \( G \) is sum divisor cordial for \( n \) is odd.

Therefore, given a positive integer \( n \), there is a sum divisor cordial graph \( G \) which has \( n \) vertices.

Example 2.1

Sum divisor cordial labeling of the graph \( G \) with 10 vertices is shown in figure 2.1.

\[
\begin{align*}
  f(v_1) &= 1, \\
  f(v_2) &= 1, \\
  f(v_3) &= 3, \\
  f(v_4) &= 1, \\
  f(v_5) &= 2, \\
  f(v_6) &= 2, \\
  f(v_7) &= 2, \\
  f(v_8) &= 3, \\
  f(v_9) &= 1, \\
  f(v_{10}) &= 2.
\end{align*}
\]

Hence, \( G \) is sum divisor cordial graph.

\[
\text{Figure 2.1}
\]
Example 2.2
The sum divisor cordial labeling of switching of a pendent vertex in path \( P_n \) is given in figure 2.2.

\[ \text{Figure 2.2} \]

Theorem 2.3
Switching of a vertex in cycle \( C_n \) admits sum divisor cordial labeling for \( n \geq 4 \).

Proof.
Let \( v_1, v_2, ..., v_n \) be the successive vertices of \( C_n \), \( G \) denotes graph is obtained by switching of vertex \( v \) of \( G = C_n \). Without loss of generality let the switched vertex be \( v_1 \).

Then \( |V(G_{v_1})| = n \) and \( |E(G_{v_1})| = 2n - 5 \).

Define \( f : V(G_{v_1}) \rightarrow \{1, 2, 3, ..., n\} \) as follows.

Case 1 : \( n \) is odd
\[
f(v) = \begin{cases} 
  i & \text{if } i = 1, 2 \pmod{4} \\
  i + 1 & \text{if } i = 3 \pmod{4} \\
  i - 1 & \text{if } i = 0 \pmod{4}
\end{cases} \quad \text{for } 1 \leq i \leq n - 1
\]
\( f(v_n) = n \)

Then \( e(0) = n - 3 \) and \( e(1) = n - 2 \), if \( n = 1 \pmod{4} \)
\( e(0) = n - 2 \) and \( e(1) = n - 3 \), if \( n = 3 \pmod{4} \)

Case 2 : \( n \) is even
\[
f(v) = \begin{cases} 
  i & \text{if } i = 1, 2 \pmod{4} \\
  i + 1 & \text{if } i = 3 \pmod{4} \\
  i - 1 & \text{if } i = 0 \pmod{4}
\end{cases} \quad \text{for } 1 \leq i \leq n
\]
Then \( e(0) = n - 3 \) and \( e(1) = n - 2 \), if \( n = 0 \pmod{4} \)
\( e(0) = n - 2 \) and \( e(1) = n - 3 \), if \( n = 2 \pmod{4} \)

In both cases, \( |e(0) - e(1)| \leq 1 \).
Hence, the obtained graph by switching of a vertex in cycle \( C_n \) is sum divisor cordial graph for \( n \geq 4 \).

Example 2.3
The sum divisor cordial labeling of switching of a vertex in cycle \( C_n \) is shown in figure 2.3.

\[ \text{Figure 2.3} \]

Theorem 2.4
\( DS(B_{n,a}) \) is sum divisor cordial graph.

Proof.
Let \( B_{n,a} \) be a graph with vertex set \( \{u, v, w, v_i, 1 \leq i \leq n\} \), where \( u, v_i \) are pendant vertices. Here, \( V(B_{n,a}) = V_1 \cup V_2 \), where \( V_1 = \{u, v_i : 1 \leq i \leq n\} \) and \( V_2 = \{u, v\} \).
In order to obtain \( DS(B_{n,a}) \) from \( B_{n,a} \), add \( w_1, w_2 \) corresponding to \( V_1 \) and \( V_2 \). Let \( G \) be a graph \( DS(B_{n,a}) \). Then, \( |V(G)| = 2n + 4 \) and \( |E(G)| = 4n + 3 \).

Define \( f : V(G) \rightarrow \{1, 2, ..., 2n + 4\} \) as follows.
\( f(u) = 1 \), \( f(v) = 3 \), \( f(w_1) = 2n + 4 \), \( f(w_2) = 2 \),
\( f(v_i) = 2i + 2 \) for \( 1 \leq i \leq n - 1 \)
\( f(v_i) = 2i + 3 \) for \( 1 \leq i \leq n - 1 \).

Then, \( e(0) = 2n + 2 \) and \( e(1) = 2n + 1 \).
In both cases, \( |e(0) - e(1)| \leq 1 \).
Hence, \( DS(B_{n,a}) \) is sum divisor cordial graph.

Example 2.4
The sum divisor cordial labeling of \( DS(B_{3,3}) \) is shown in figure 2.4.

\[ \text{Figure 2.4} \]

Theorem 2.5
Let \( G \) be any sum divisor cordial graph of order \( p \) and \( K_{2,n} \) be a bipartite graph with the bipartition \( V = V_1 \cup V_2 \) with \( V_1 = \{v_1, v_2\} \) and \( V_2 = \{u_1, u_2, ..., u_n\} \).
Then the graph \( G * K_{2,n} \) obtained by identifying the vertices \( v_1 \) and \( v_2 \) of \( K_{2,n} \) with that labeled any one odd number and any one even number respectively in \( G \) is also sum divisor cordial graph.

Proof.
Let \( G \) be any sum divisor cordial graph of order \( p \). Then in \( G, |e(0) - e(1)| \leq 1 \).
Consider any two vertices of \( G \) having one odd label and one even label. Without loss of generality, let \( w_i \) and \( w_j \) be the consider vertices with the labels 1 and 2. Let \( V = V_1 \cup V_2 \) be the bipartition of \( K_{2,n} \) such that \( V_1 = \{v_1, v_2\} \) and \( V_2 = \{u_1, u_2, ..., u_n\} \).
Now assign the labels \( p + 1, p + 2, ..., p + n \) to the vertices \( u_1, u_2, ..., u_n \) respectively and identify the vertices \( v_1 \) and \( v_2 \) of \( K_{2,n} \) with that labeled 1 and 2 respectively in \( G \).
The edges of \( K_{2,n} \) incident with the vertex \( w_i \) have the label 1 and with the vertex \( w_j \) have the label 2. Then, the edges of \( K_{2,n} \) contribute equal numbers, namely \( n \) to both \( e(1) \) and \( e(0) \) in \( G * K_{2,n} \).
Thus, \( |e(0) - e(1)| \leq 1 \) in \( G * K_{2,n} \).
Hence \( G * K_{2,n} \) is sum divisor cordial.
Example 2.5

Sum divisor cordial labeling of the graph G and G*K_{3,5} are shown in figure 2.5.

Figure 2.5

Theorem 2.6

Let G be any sum divisor cordial graph of order p and K_{3,n} be a bipartite graph with the bipartition V = V_1 \cup V_2 with V_1 = \{x_1,x_2,x_3\} and V_2 = \{y_1,y_2,...,y_n\}, where n is even. Then the graph G*K_{3,n} obtained by identifying the vertices x_1, x_2 and x_3 of K_{3,n} with that labeled any two odd number and any one even number or any one odd number and any two even number respectively in G is also sum divisor cordial graph.

Proof.

Let G be any sum divisor cordial graph of order p. Then in G, \(|e(0) - e(1)| \leq 1.

Let V = V_1 \cup V_2 be the bipartition of K_{3,n} such that V_1 = \{x_1,x_2,x_3\} and V_2 = \{y_1,y_2,...,y_n\}.

Case 1: In G, e(0) = e(1).
Consider any three vertices of G having one odd label and two even labels.
Without loss of generality, let v_i, v_j and v_k be the vertices having the labels 1, 2 and 4.
Now assign the labels p+1,p+2,...,p+n to the vertices y_1,y_2,...,y_n respectively and identify the vertices x_1, x_2 and x_3 of K_{3,5} with that labeled 1, labeled 2 and labeled 4 respectively in G.

Subcase 1(a): p is odd or even and n is even.
Then, the edges of K_{3,n} incident with the vertices v_i, v_j and v_k contribute \(\frac{3n-1}{2}\) numbers of 0’s and \(\frac{3n+1}{2}\) numbers of 1’s in G*K_{3,n}.

Thus, \(|e(0) - e(1)| \leq 1\) in G*K_{3,n}.

Subcase 1(b): p is even and n is odd.
Then, the edges of K_{3,n} incident with the vertices v_i, v_j and v_k contribute \(\frac{3n+1}{2}\) numbers of 0’s and \(\frac{3n-1}{2}\) numbers of 1’s in G*K_{3,n}.

Thus, \(|e(0) - e(1)| \leq 1\) in G*K_{3,n}.

Subcase 1(c): p is odd and n is odd.
Then, the edges of K_{3,n} incident with the vertices v_i, v_j and v_k contribute \(\frac{3n+1}{2}\) numbers of 0’s and \(\frac{3n-1}{2}\) numbers of 1’s in G*K_{3,n}.

Thus, \(|e(0) - e(1)| \leq 1\) in G*K_{3,n}.

Case 2: In G, e(0) = e(1) + 1.

Subcase 2(a): p is odd or even and n is even.
Consider any three vertices of G having one odd label and two even labels.
Without loss of generality, let v_i, v_j and v_k be the vertices having the labels 1, 2 and 4.
Now assign the labels p+1,p+2,...,p+n to the vertices y_1,y_2,...,y_n respectively and identify the vertices x_1, x_2 and x_3 of K_{3,n} with that labeled 1, labeled 2 and labeled 4 respectively in G.
Then, the edges of K_{3,n} incident with the vertices v_i, v_j and v_k contribute equal numbers of 0’s and 1’s in G*K_{3,n}.

Thus, \(|e(0) - e(1)| \leq 1\) in G*K_{3,n}.

Subcase 2(b): p is even and n is odd.
Consider any three vertices of G having two odd labels and one even label.
Without loss of generality, let v_i, v_j and v_k be the vertices having the labels 1, 2 and 3.
Now assign the labels p+1,p+2,...,p+n to the vertices y_1,y_2,...,y_n respectively and identify the vertices x_1, x_2 and x_3 of K_{3,n} with that labeled 1, labeled 2 and labeled 3 respectively in G.

Then, the edges of K_{3,n} incident with the vertices v_i, v_j and v_k contribute \(\frac{3n-1}{2}\) numbers of 0’s and \(\frac{3n+1}{2}\) numbers of 1’s in G*K_{3,n}.

Thus, \(|e(0) - e(1)| = 0\) in G*K_{3,n}.

Subcase 2(c): p is odd and n is odd.
Consider any three vertices of G having one odd label and two even labels.
Without loss of generality, let v_i, v_j and v_k be the vertices having the labels 1, 2 and 4.
Now assign the labels p+1,p+2,...,p+n to the vertices y_1,y_2,...,y_n respectively and identify the vertices x_1, x_2 and x_3 of K_{3,n} with that labeled 1, labeled 2 and labeled 4 respectively in G.

Then, the edges of K_{3,n} incident with the vertices v_i, v_j and v_k contribute \(\frac{3n-1}{2}\) numbers of 0’s and \(\frac{3n+1}{2}\) numbers of 1’s in G*K_{3,n}.

Thus, \(|e(0) - e(1)| = 0\) in G*K_{3,n}.

Case 3: In G, e(0) + 1 = e(1).

Subcase 3(a): p is odd or even and n is even.
Consider any three vertices of G having one odd label and two even labels.
Without loss of generality, let v_i, v_j and v_k be the vertices having the labels 1, 2 and 4.
Now assign the labels p+1,p+2,...,p+n to the vertices y_1,y_2,...,y_n respectively and identify the vertices x_1, x_2 and x_3 of K_{3,n} with that labeled 1, labeled 2 and labeled 4 respectively in G.

Then, the edges of K_{3,n} incident with the vertices v_i, v_j and v_k contribute equal numbers of 0’s and 1’s in G*K_{3,n}.

Thus, \(|e(0) - e(1)| \leq 1\) in G*K_{3,n}.
Subcase 3(b): p is even and n is odd.

Consider any three vertices of G having one odd label and two even labels.

Without loss of generality, let \(v_i, v_j\) and \(v_k\) be the vertices having the labels 1, 2 and 4.

Now assign the labels \(p+1, p+2, \ldots, p+n\) to the vertices \(y_1, y_2, \ldots, y_n\) respectively and identify the vertices \(x_1, x_2\) and \(x_3\) of \(K_{3,n}\) with that labeled 1, labeled 2 and labeled 4 respectively in G.

Then, the edges of \(K_{3,n}\) incident with the vertices \(v_i, v_j\) and \(v_k\) contribute \(\frac{3n+1}{2}\) numbers of 0’s and \(\frac{3n-1}{2}\) numbers of 1’s in \(G \ast K_{3,n}\).

Thus, \(|e_f(0) - e_f(1)| = 0\) in \(G \ast K_{3,n}\).

Subcase 3(c): p is odd and n is odd.

Consider any three vertices of G having two odd labels and one even label.

Without loss of generality, let \(v_i, v_j\) and \(v_k\) be the vertices having the labels 1, 2 and 3.

Now assign the labels \(p+1, p+2, \ldots, p+n\) to the vertices \(y_1, y_2, \ldots, y_n\) respectively and identify the vertices \(x_1, x_2\) and \(x_3\) of \(K_{3,n}\) with that labeled 1, labeled 2 and labeled 3 respectively in G.

Then, the edges of \(K_{3,n}\) incident with the vertices \(v_i, v_j\) and \(v_k\) contribute \(\frac{3n+1}{2}\) numbers of 0’s and \(\frac{3n-1}{2}\) numbers of 1’s in \(G \ast K_{3,n}\).

Thus, \(|e_f(0) - e_f(1)| = 0\) in \(G \ast K_{3,n}\).

Hence \(G \ast K_{3,n}\) is sum divisor cordial.

Example 2.6
Sum divisor cordial labeling of the graph G and \(G \ast K_{3,n}\) are shown in figure 2.6.

Figure 2.6

III. Conclusions

In this paper, we prove that the switching of a pendant vertex in path \(P_n\), switching of any vertex in cycle \(C_n, DS(B_{a,n}), G \ast K_{2,n}\) and \(G \ast K_{3,n}\) are sum divisor cordial graph.

REFERENCES