Common Coupled Fixed Point Results for Generalized Rational Type Contractions in Complex Valued Metric Spaces

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Abstract

In this paper, we prove a common coupled fixed point results for generalized rational type contractions in complex valued metric spaces which generalize common coupled fixed point theorems due to Marwan Amin Kutbi et al.,\cite{1}.

Key words: Complex valued metric space; Coupled fixed point; Common coupled fixed point; Coupled coincidence point;

1 Introduction

Azam et al.\cite{2} introduced new spaces called complex valued metric spaces and established the existence of fixed point theorems under the contraction condition. Subsequently, Rouzkard and Imdad \cite{3} established some common fixed point theorems satisfying certain rational expressions in complex valued metric spaces which generalize, unify and complement the results of Azam et al.\cite{2}, Sintunavarat and Kumam \cite{4} obtained common fixed point results by replacing constant of contractive condition to control functions. Recently, Klin-eam and Suanoom \cite{5} extend the concept of complex valued metric spaces and generalized the results of Azam et al.\cite{2} and Rouzkard and Imdad \cite{3}.
The concept of coupled fixed point was first introduced by Bhaskar and Laxikantham [10] in 2006. Recently some researchers prove some coupled fixed point theorems in complex valued metric space in [11],[12]. In [1], Marwan Amin Kutbi et al., gave a common coupled fixed point results for generalized contraction in complex valued metric space and proved following theorem.

**Theorem 1.1.** Let \((X,d)\) be a complete complex valued metric space, and let the mappings \(S,T: X \times X \to X\) satisfy

\[
d(S(x,y), T(u,v)) \leq \frac{\alpha d(x,u) + d(y,v)}{2} + \frac{\beta d(x,S(x,y)T(u,v)) + \gamma d(u,S(x,y)d(x,T(u,v))}{1 + d(x,u) + d(y,v)}
\]

for all \(x,y,u,v \in X\) and \(\alpha, \beta, \gamma\) are non negative real with \(\alpha + \beta + \gamma < 1\). Then \(S\) and \(T\) have a unique common coupled fixed point.

**Theorem 1.2.** Let \((X,d)\) be a complete complex valued metric space, and let the mappings \(S,T: X \times X \to X\) satisfy

\[
d(S(x,y), T(u,v)) \leq \begin{cases} 
\frac{\alpha d(x,u) + d(y,v)}{2} + \frac{\beta d(x,S(x,y)T(u,v)) + \gamma d(u,S(x,y)d(x,T(u,v))}{1 + d(x,u) + d(y,v)} & \text{if } D \neq 0 \\
0 & \text{if } D = 0.
\end{cases}
\]

for all \(x,y,u,v \in X\), where \(D = d(x,T(u,v)) + d(u,S(x,y)) + d(x,u) + d(y,v)\) and \(\alpha, \beta\) are nonnegative reals with \(\alpha + \beta < 1\). Then \(S\) and \(T\) have a unique common coupled fixed point.

## 2 Preliminaries

Let \(\mathbb{C}\) be the set of complex numbers and \(z_1, z_2 \in \mathbb{C}\). Define a partial order \(\preceq\) on \(\mathbb{C}\) as follows:

\[z_1 \preceq z_2 \text{ iff } \Re(z_1) \leq \Re(z_2), \quad \Im(z_1) \leq \Im(z_2).\]

Note that \(0 \preceq z_1\) and \(z_1 \neq z_2, z_1 \preceq z_2\) implies \(|z_1| < |z_2|\).

The following definition is recently introduced by Azam et al. [2]

**Definition 2.1.** Let \(X\) be a non empty set. Suppose that the mapping \(d: X \times X \to \mathbb{C}\) satisfies the following conditions:

1. \(0 \preceq d(x,y)\) for all \(x,y \in X\) and \(d(x,y) = 0\) if and only if \(x = y\);
2. \(d(x,y) = d(y,x)\) for all \(x,y \in X\);
3. \(d(x,y) \preceq d(x,z) + d(z,y)\) for all \(x,y,z \in X\).
Then, $d$ is called a complex valued metric on $X$, and $(X, d)$ is called a complex valued metric space.

**Example 2.1.** Let $X = \mathbb{C}$. Define the mapping $d : X \times X \to \mathbb{C}$ by

$$d(x, y) = i|x - y|, \quad \forall \quad x, y \in X.$$ 

Then, $(X, d)$ is a complex valued metric space.

**Definition 2.2.** Let $(X, d)$ be a complex valued metric space.

1. A point $x \in X$ is called interior point of a set $A \subseteq X$ whenever there exists $0 < r \in \mathbb{C}$ such that $B(x, r) := \{y \in X : d(x, y) < r\} \subseteq A$.
2. A point $x \in X$ is called a limit point of a set $A$ whenever for every $0 < r \in \mathbb{C}$, $B(x, r) \cap (A - \{x\}) \neq \emptyset$.
3. A subset $A \subseteq X$ is called open whenever each element of $A$ is an interior point of $A$.
4. A subset $A \subseteq X$ is called closed whenever each element of $A$ belongs to $A$.
5. A subset $A \subseteq X$ is called closed whenever each element of $A$ belongs to $A$.
6. A sub-basis for a Hausdorff topology $\tau$ on $X$ is a family $F := \{B(x, r) : x \in X$ and $0 < r\}$.

**Definition 2.3.** Let $(X, d)$ be a complex valued metric space. A sequence $\{x_n\}$ in $X$ is said to be

1. convergent to $x$, if for every $c \in \mathbb{C}$ with $0 < c$ there is $k \in \mathbb{N}$ such that, for all $n > k, d(x_n, x) < c$, we denote this by $\{x_n\} \to x$ as $n \to \infty$ or $\lim_{n \to \infty} x_n = x$;
2. Cauchy, if for every $c \in \mathbb{C}$ with $0 < c$ there is $k \in \mathbb{N}$ such that, for all $n > k, d(x_n, x_{n+m}) < c$, where $m \in \mathbb{N}$;
3. complete, if every Cauchy sequence in $X$ converges in $X$.

In [2], Azam et al. established the following two lemmas.

**Lemma 2.2.** [2] Let $(X, d)$ be a complex valued metric space, and let $\{x_n\}$ be a sequence in $X$. Then, $\{x_n\}$ converges to $x$ if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$.

**Lemma 2.3.** [2] Let $(X, d)$ be a complex valued metric space, and let $\{x_n\}$ be a sequence in $X$. Then, $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$, where $m \in \mathbb{N}$.

**Definition 2.4.** [10] An element $(x, y) \in X \times X$ is called a coupled fixed point of $T : X \times X \to X$ if

$$x = T(x, y) \quad y = T(y, x)$$
Definition 2.5. [1] An element \(d(x, y) \in X \times X\) is called a coupled coincidence point of \(S, T: X \times X \to X\) if
\[ S(x, y) = T(x, y), \quad S(y, x) = T(y, x). \]

Example 2.4. An element \((x, y) \in X \times X\) is called a coupled coincidence point of \(S, T: X \times X \to X\) defined as \(S(x, y) = x^2y^2\) and \(T(x, y) = \left(\frac{1}{2}(x + y)\right)^2\) for all \(x, y \in X\). Then \((0, 0), (1, 2),\) and \((2, 1)\) are coupled coincidence points of \(S\) and \(T\).

Example 2.5. Let \(X = \mathbb{R}\) and \(S, T: X \times X \to X\) defined as \(S(x, y) = x + y + \sin(x + y)\) and \(T(x, y) = x + y + xy + \cos(x + y)\) for all \(x, y \in X\). Then \((0, \frac{\pi}{4})\) and \((\frac{\pi}{4}, 0)\) are coupled coincidence points of \(S\) and \(T\).

Definition 2.6. [1] An element \((x, y) \in X \times X\) is called a common coupled fixed point of \(S, T: X \times X \to X\) if
\[ x = S(x, y) = T(x, y) \quad y = S(y, x) = T(y, x). \]

Example 2.6. Let \(X = \mathbb{R}\) and \(S, T: X \times X \to X\) defined as \(S(x, y) = x(x + (y-1)^2)\) and \(T(x, y) = x(\sqrt{x^2 + y^2} + 4 - 2)\) for all \(x, y \in X\). Then \((0, 0), (1, 2)\) and \((2, 1)\) are common coupled fixed points of \(S\) and \(T\).

The purpose of this paper is to generalize the theorem of Marwan Amin Kutbi et al.,[1] to common coupled fixed point results for generalized rational type contractions in complex valued metric Spaces.
3 Main Results

Theorem 3.1. Let \((X,d)\) be a complete complex valued metric space, and let the mappings \(S,T : X \times X \to X\) satisfy

\[
d(S(x,y), T(u,v)) \leq a_1 \frac{d(x,u) + d(y,v)}{2} + a_2 \frac{d(x,S(x,y))d(u,T(u,v))}{1 + d(x,u) + d(y,v) + d(u,S(x,y))} + a_3 \frac{d(x,u) + d(y,v) + d(u,S(x,y))}{d(u,S(x,y))d(x,u)} + a_4 \frac{d(S(x,y), T(u,v))d(y,v)}{1 + d(x,u) + d(y,v) + d(u,S(x,y))} + a_5 \frac{d(x,u) + d(y,v) + d(u,S(x,y))}{d(u,S(x,y))d(x,u)} + a_6 \frac{d(u,T(u,v))d(y,v)}{1 + d(x,u) + d(y,v) + d(u,S(x,y))} + a_7 \frac{d(u,S(x,y))d(x,u)}{1 + d(x,u) + d(y,v) + d(u,S(x,y))} + a_8 \frac{d(u,S(x,y))d(y,v)}{1 + d(x,u) + d(y,v) + d(u,S(x,y))} + a_9 \max\{d(u,S(x,y)), d(S(x,y), T(u,v))\}
\]

for all \(x, y, u, v \in X\) and \(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9 \geq 0\) with \(a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 < 1\) and \(a_1 + a_3 + a_4 + a_5 + a_7 + a_8 + a_9 < 1\). Then \(S\) and \(T\) have a unique common coupled fixed point.

Proof. Let \(x_0\) and \(y_0\) be arbitrary points in \(X\). Define \(x_{2k+1} = S(x_{2k}, y_{2k}), y_{2k+1} = S(y_{2k}, x_{2k})\) and \(x_{2k+2} = T(x_{2k+1}, y_{2k+1}), y_{2k+2} = T(y_{2k+1}, x_{2k+1})\) for \(k = 0, 1, \ldots\)
Then,

\[ d(x_{2k+1}, x_{2k+2}) = d(S(x_{2k}, y_{2k}), T(x_{2k+1}, y_{2k+1})) \]

\[ \leq a_1 d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1}) \]

\[ + \frac{a_2}{2} \left( d(x_{2k}, S(x_{2k}, y_{2k}))d(x_{2k+1}, T(x_{2k+1}, y_{2k+1})) + d(x_{2k+1}, S(x_{2k}, y_{2k}))d(x_{2k}, T(x_{2k+1}, y_{2k+1})) \right) \]

\[ + a_3 d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1}) + d(x_{2k+1}, S(x_{2k}, y_{2k})) \]

\[ + \frac{a_4}{2} \left( d(x_{2k}, S(x_{2k}, y_{2k}))d(x_{2k+1}, T(x_{2k+1}, y_{2k+1})) + d(x_{2k+1}, S(x_{2k}, y_{2k}))d(x_{2k}, T(x_{2k+1}, y_{2k+1})) \right) \]

\[ + a_5 d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1}) + d(x_{2k+1}, S(x_{2k}, y_{2k})) \]

\[ + \frac{a_6}{2} \left( d(x_{2k+1}, T(x_{2k+1}, y_{2k+1}))d(y_{2k}, y_{2k+1}) \right) \]

\[ + a_7 d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1}) + d(x_{2k+1}, S(x_{2k}, y_{2k})) \]

\[ + \frac{a_8}{2} \left( d(x_{2k+1}, T(x_{2k+1}, y_{2k+1}))d(y_{2k}, y_{2k+1}) \right) \]

\[ + a_9 \max \{ d(x_{2k+1}, S(x_{2k}, y_{2k})), d(S(x_{2k}, y_{2k}), T(x_{2k+1}, y_{2k+1})) \} \]

\[ \leq a_1 d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1}) \]

\[ + \frac{a_2}{2} \left( d(x_{2k}, S(x_{2k}, y_{2k}))d(x_{2k+1}, x_{2k+2}) + d(x_{2k+1}, S(x_{2k}, y_{2k}))d(x_{2k}, x_{2k+2}) \right) \]

\[ + a_3 d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1}) + d(x_{2k+1}, S(x_{2k}, y_{2k})) \]

\[ + \frac{a_4}{2} \left( d(x_{2k}, S(x_{2k}, y_{2k}))d(x_{2k+1}, x_{2k+2}) + d(x_{2k+1}, S(x_{2k}, y_{2k}))d(x_{2k}, x_{2k+2}) \right) \]

\[ + a_5 d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1}) + d(x_{2k+1}, S(x_{2k}, y_{2k})) \]

\[ + \frac{a_6}{2} \left( d(x_{2k}, S(x_{2k}, y_{2k}))d(x_{2k+1}, x_{2k+2}) + d(x_{2k+1}, S(x_{2k}, y_{2k}))d(x_{2k}, x_{2k+2}) \right) \]

\[ + a_7 d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1}) + d(x_{2k+1}, S(x_{2k}, y_{2k})) \]

\[ + \frac{a_8}{2} \left( d(x_{2k}, S(x_{2k}, y_{2k}))d(x_{2k+1}, x_{2k+2}) + d(x_{2k+1}, S(x_{2k}, y_{2k}))d(x_{2k}, x_{2k+2}) \right) \]

\[ + a_9 \max \{ d(x_{2k+1}, x_{2k+1}), d(x_{2k+1}, x_{2k+2}) \} \]
which implies that

\[
(1 - (a_2 + a_4 + a_5 + a_6 + a_9))d(x_{2k+1}, x_{2k+2}) \leq a_1 \frac{d(x_{2k}, x_{2k+1})}{2} + a_1 \frac{d(y_{2k}, y_{2k+1})}{2}
\]

\[
|d(x_{2k+1}, x_{2k+2})| \leq a_1 \frac{|d(x_{2k}, x_{2k+1})|}{2(1 - (a_2 + a_4 + a_5 + a_6 + a_9))} + a_1 \frac{|d(y_{2k}, y_{2k+1})|}{2(1 - (a_2 + a_4 + a_5 + a_6 + a_9))} \tag{1}
\]

Proceeding similarly one can prove that

\[
|d(y_{2k+1}, y_{2k+2})| \leq a_1 \frac{|d(x_{2k}, x_{2k+1})|}{2(1 - (a_2 + a_4 + a_5 + a_6 + a_9))} + a_1 \frac{|d(y_{2k}, y_{2k+1})|}{2(1 - (a_2 + a_4 + a_5 + a_6 + a_9))} \tag{2}
\]

Adding (1) and (2), we get

\[
|d(x_{2k+1}, x_{2k+2})| + |d(y_{2k+1}, y_{2k+2})| \leq \frac{a_1}{(1 - (a_2 + a_4 + a_5 + a_6 + a_9))} \left[ |d(x_{2k}, x_{2k+1})| + |d(y_{2k}, y_{2k+1})| \right]
\]

where \( k = \frac{a_1}{1 - (a_2 + a_4 + a_5 + a_6 + a_9)} < 1 \).

Also, \( a_1 \),

\[
|d(x_{2k+2}, x_{2k+3})| \leq a_1 \frac{|d(x_{2k+1}, x_{2k+2})|}{2(1 - (a_2 + a_4 + a_5 + a_6 + a_9))} + a_1 \frac{|d(y_{2k+1}, y_{2k+2})|}{2(1 - (a_2 + a_4 + a_5 + a_6 + a_9))} \tag{3}
\]

\[
|d(y_{2k+2}, y_{2k+3})| \leq a_1 \frac{|d(y_{2k+1}, y_{2k+2})|}{2(1 - (a_2 + a_4 + a_5 + a_6 + a_9))} + a_1 \frac{|d(x_{2k+1}, x_{2k+2})|}{2(1 - (a_2 + a_4 + a_5 + a_6 + a_9))} \tag{4}
\]

Adding (3) and (5), we get

\[
|d(x_{2k+2}, x_{2k+3})| + |d(y_{2k+2}, y_{2k+3})| \leq \frac{a_1}{(1 - (a_2 + a_4 + a_5 + a_6 + a_9))} \left[ |d(x_{2k+1}, x_{2k+2})| + |d(y_{2k+1}, y_{2k+2})| \right]
\]

\[
= k |d(x_{2k+1}, x_{2k+2})| + |d(y_{2k+1}, y_{2k+2})| \]

\[
\leq k^2 |d(x_{2k}, x_{2k+1})| + |d(y_{2k}, y_{2k+1})|.
\]
Continuing this way, we have

\[ |d(x_n, x_{n+1})| + |d(y_n, y_{n+1})| \leq k|d(x_{n-1}, x_n)| + |d(y_{n-1}, y_n)| \]
\[ \leq k^2|d(x_{n-2}, x_{n-1})| + |d(y_{n-2}, y_{n-1})| \]
\[ \leq \cdots \leq k^n|d(x_0, x_1)| + |d(y_0, y_1)| \]

Now if \(|d(x_n, x_{n+1})| + |d(y_n, y_{n+1})| = \delta_n\), then

\[ \delta_n \leq k\delta_{n-1} \leq \cdots \leq k^n \delta_0. \]

Without loss of generality, we take \(m > n\). Since \(0 \leq k < 1\), so we get

\[ |d(x_n, x_m)| + |d(y_n, y_m)| \leq |d(x_n, x_{n+1})| + |d(y_n, y_{n+1})| \]
\[ + |d(x_{n+1}, x_{n+2})| + |d(y_{n+1}, y_{n+2})| + \cdots \]
\[ + |d(x_{m-1}, x_m)| + |d(y_{m-1}, y_m)| \]
\[ \leq [k^n \delta_0 + k^{n+1} \delta_0 + \cdots + k^{m-1} \delta_0] \]
\[ \leq k^n[1 + k + k^2 + \cdots] \delta_0 \]
\[ = \frac{k^n}{1-k} \delta_0 \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty. \]

This implies that \(\{x_n\}\) and \(\{y_n\}\) are Cauchy sequence in \(X\). Since \(X\) is complete, there exists \(x, y \in X\) such that \(x_n \rightarrow x\) and \(y_n \rightarrow y\) as \(n \rightarrow \infty\). We now show that \(x = S(x, y)\) and \(y = S(y, x)\). We suppose on the contrary that \(x \neq S(x, y)\) and \(y \neq S(y, x)\) so that \(0 < d(x, S(x, y)) = l_1\) and \(0 < d(y, S(y, x)) = l_2\); we
would then have

\[ l_1 = d(x, S(x, y)) \leq d(x, x_{2k+2}) + d(x_{2k+2}, S(x, y)) \]
\[ \leq d(x, x_{2k+2}) + d(S(x, y), T(x_{2k+1}, y_{2k+1})) \]
\[ \leq d(x, x_{2k+2}) + a_1 d(x, x_{2k+1}) + a_2 d(x, S(x, y)) + a_3 d(x_{2k+1}, S(x, y)) d(x, x_{2k+1}) + a_4 d(S(x, y), T(x_{2k+1}, y_{2k+1})) d(x, x_{2k+1}) + a_5 d(x_{2k+1}, T(x_{2k+1}, y_{2k+1})) d(y, y_{2k+1}) + a_6 d(x, x_{2k+1}) + d(y, y_{2k+1}) + d(x_{2k+1}, S(x, y)) d(x, x_{2k+1}) + a_7 d(x_{2k+1}, S(x, y)) d(y, y_{2k+1}) + a_8 d(x, x_{2k+1}) + d(y, y_{2k+1}) + d(x_{2k+1}, S(x, y)) d(x_{2k+1}, x_{2k+2}) + a_9 d(x, S(x, y)) d(x_{2k+1}, x_{2k+2}) + a_2 d(x, x_{2k+1}) + d(y, y_{2k+1}) + d(x_{2k+1}, S(x, y)) d(x_{2k+1}, x_{2k+2}) + a_3 d(x, x_{2k+1}) + d(y, y_{2k+1}) + d(x_{2k+1}, S(x, y)) d(x_{2k+1}, x_{2k+2}) + a_4 d(S(x, y), x_{2k+2}) d(x, y_{2k+1}) + a_5 d(x, x_{2k+1}) + d(y, y_{2k+1}) + d(x_{2k+1}, S(x, y)) d(x_{2k+1}, x_{2k+2}) + a_6 d(x, x_{2k+1}) + d(y, y_{2k+1}) + d(x_{2k+1}, S(x, y)) d(x_{2k+1}, x_{2k+2}) + a_7 d(x, x_{2k+1}) + d(y, y_{2k+1}) + d(x_{2k+1}, S(x, y)) d(x_{2k+1}, x_{2k+2}) + a_8 d(x, x_{2k+1}) + d(y, y_{2k+1}) + d(x_{2k+1}, S(x, y)) d(x_{2k+1}, x_{2k+2}) + a_9 \max\{d(x_{2k+1}, x_{2k+1}), d(x_{2k+1}, x_{2k+2})\} \]
which implies that

\[ |l_1| \leq \frac{|d(x, x_{2k+2})| + a_1 |d(x, x_{2k+1})| + |d(y, y_{2k+1})|}{2} \\
+ a_2 \left[ |1 + d(x, x_{2k+1}) + d(y, y_{2k+1}) + d(x_{2k+1}, S(x, y))| \right] \\
+ a_3 \left[ |d(x_{2k+1}, x_{2k+2})| |d(x, x_{2k+1})| \right] \\
+ a_4 \left[ |1 + d(x, x_{2k+1}) + d(y, y_{2k+1}) + d(x_{2k+1}, S(x, y))| \right] \\
+ a_5 \left[ |d(S(x, y), x_{2k+2})| |d(y, y_{2k+1})| \right] \\
+ a_6 \left[ |1 + d(x, x_{2k+1}) + d(y, y_{2k+1}) + d(x_{2k+1}, S(x, y))| \right] \\
+ a_7 \left[ |d(x_{2k+1}, S(x, y))| |d(x, x_{2k+1})| \right] \\
+ a_8 \left[ |d(x_{2k+1}, S(x, y))| |d(y, y_{2k+1})| \right] \\
+ a_9 \max \{ |d(x_{2k+1}, x_{2k+2})|, |d(x_{2k+1}, x_{2k+2})| \} \]

Since \( \{x_n\} \) and \( \{y_n\} \) are convergent to \( x \) and \( y \), therefore by taking limit as \( k \to \infty \) we get \( |l_1| \leq 0 \). Which is contradiction, so \( |d(x, S(x, y))| = 0 \Rightarrow x = S(x, y) \).

Similarly we can prove that \( y = S(x, y) \). Also we can prove that \( x = T(x, y) \) and \( y = T(y, x) \). Thus \( (x, y) \) is a common coupled fixed point of \( S \) and \( T \).

We now show that \( S \) and \( T \) have a unique common coupled fixed point. For this, assume that \( (x, y) \in X \times X \) is a second common coupled fixed point of \( S \) and \( T \). Then
\[ d(x, x_1) = d(S(x, y), T(x_1, y_1)) \]
\[ \leq a_1 \frac{d(x, x_1) + d(y, y_1)}{2} \]
\[ + a_2 \frac{d(x, S(x, y))d(x_1, T(x_1, y_1))}{1 + d(x, x_1) + d(y, y_1) + d(x_1, S(x, y))} \]
\[ + a_3 \frac{d(x_1, S(x, y))d(x, T(x_1, y_1))}{1 + d(x, x_1) + d(y, y_1) + d(x_1, S(x, y))} \]
\[ + a_4 \frac{d(S(x, y), T(x_1, y_1))d(x, x_1)}{1 + d(x, x_1) + d(y, y_1) + d(x_1, S(x, y))} \]
\[ + a_5 \frac{d(S(x, y), T(x_1, y_1))d(y, y_1)}{1 + d(x, x_1) + d(y, y_1) + d(x_1, S(x, y))} \]
\[ + a_6 \frac{d(x_1, S(x, y))d(x, x_1)}{1 + d(x, x_1) + d(y, y_1) + d(x_1, S(x, y))} \]
\[ + a_7 \frac{d(x_1, S(x, y))d(y, y_1)}{1 + d(x, x_1) + d(y, y_1) + d(x_1, S(x, y))} \]
\[ + a_8 \frac{d(x, x_1) + d(y, y_1) + d(x_1, S(x, y))}{a_9 \max \{d(x_1, S(x, y)), d(S(x, y), T(x_1, y_1))\}} \]
\[ = a_1 \frac{d(x, x_1) + d(y, y_1)}{2} \]
\[ + a_2 \frac{d(x, x_1) + d(y, y_1) + d(x_1, x)}{d(x_1, x)d(x, x_1)} \]
\[ + a_3 \frac{d(x, x_1) + d(y, y_1) + d(x_1, x)}{d(x_1, x)d(x, x_1)} \]
\[ + a_4 \frac{d(x_1, x) + d(y, y_1) + d(x_1, x)}{d(x_1, x)d(y, y_1)} \]
\[ + a_5 \frac{d(x, x_1) + d(y, y_1) + d(x_1, x)}{d(y, y_1)d(x_1, x_1)} \]
\[ + a_6 \frac{d(x, x_1) + d(y, y_1) + d(x_1, x)}{d(y, y_1)d(x, x_1)} \]
\[ + a_7 \frac{d(x, x_1) + d(y, y_1) + d(x_1, x)}{d(x, x_1)d(y, y_1)} \]
\[ + a_8 \frac{d(x, x_1) + d(y, y_1) + d(x_1, x)}{d(x, x_1)d(y, y_1)} \]
\[ + a_9 \max \{d(x_1, x), d(x, x_1)\} \]
Thus

\[ |d(x, x_1)| \leq a_1 \frac{|d(x, x_1)|}{2} + a_3 \frac{|d(x, x)|d(x, x_1)}{[1 + 2d(x, x_1) + d(y, y_1)]} + a_4 \frac{|d(x, x_1)|d(x, x_1)}{[1 + 2d(x, x_1) + d(y, y_1)]} + a_5 \frac{|d(x, x_1)|d(x, x_1)}{[1 + 2d(x, x_1) + d(y, y_1)]} + a_7 \frac{|d(x, x_1)|d(x, x_1)}{[1 + 2d(x, x_1) + d(y, y_1)]} + a_8 \frac{|d(x, x_1)|d(x, x_1)}{[1 + 2d(x, x_1) + d(y, y_1)]} + a_9 |d(x, x_1)| \]

Since \( |1 + 2d(x, x_1) + d(y, y_1)| \geq |d(x, x_1)| \), so we get

\[ (1 - \frac{a_1}{2} - a_3 - a_4 - a_5 - a_7 - a_8 - a_9) |d(x, x_1)| \leq a_1 \frac{|d(y, y_1)|}{2} \]

\[ |d(x, x_1)| \leq \frac{a_1}{(2 - a_1 - 2a_3 - 2a_4 - 2a_5 - 2a_7 - 2a_8 - 2a_9)} |d(y, y_1)| \quad (5) \]

Similarly,

\[ |d(y, y_1)| \leq \frac{a_1}{(2 - a_1 - 2a_3 - 2a_4 - 2a_5 - 2a_7 - 2a_8 - 2a_9)} |d(x, x_1)| \quad (6) \]

Adding (5) and (6), we get

\[ |d(x, x_1)| + |d(y, y_1)| \leq \frac{a_1}{(2 - a_1 - 2a_3 - 2a_4 - 2a_5 - 2a_7 - 2a_8 - 2a_9)} [|d(y, y_1)| + |d(x, x_1)|] \]

\[ 1 - \frac{a_1}{(2 - a_1 - 2a_3 - 2a_4 - 2a_5 - 2a_7 - 2a_8 - 2a_9)} [||d(y, y_1)| + |d(x, x_1)||] \leq 0 \]

\[ \frac{2(1 - a_1 - a_3 - a_4 - a_5 - a_7 - a_8 - a_9)}{(2 - a_1 - 2a_3 - 2a_4 - 2a_5 - 2a_7 - 2a_8 - 2a_9)} [||d(y, y_1)| + |d(x, x_1)||] \leq 0. \]

Since \( a_1 + a_3 + a_4 + a_5 + a_7 + a_8 + a_9 < 1 \).

Therefore

\[ \frac{2(1 - a_1 - a_3 - a_4 - a_5 - a_7 - a_8 - a_9)}{(2 - a_1 - 2a_3 - 2a_4 - 2a_5 - 2a_7 - 2a_8 - 2a_9)} > 0 \]

Hence

\[ ||d(y, y_1)| + |d(x, x_1)|| \leq 0. \]

Which implies that \( x = x_1 \) and \( y = y_1 \) \( \Rightarrow (x, y) = (x_1, y_1) \).

Thus, \( S \) and \( T \) have unique common coupled fixed point.
Corollary 3.2. Let \((X, d)\) be a complete complex valued metric space, and let the mappings \(T: X \times X \to X\) satisfy

\[
d(T(x, y), T(u, v)) \leq a_1 \frac{d(x, u) + d(y, v)}{2} + a_2 \frac{d(x, T(x, y))d(u, T(u, v))}{1 + d(x, u) + d(y, v) + d(u, T(x, y))}
+ a_3 \frac{d(x, T(x, y))d(x, T(u, v))}{1 + d(x, u) + d(y, v) + d(u, T(x, y))}
+ a_4 \frac{d(T(x, y), T(u, v))d(x, u)}{1 + d(x, u) + d(y, v) + d(u, T(x, y))}
+ a_5 \frac{d(T(x, y), T(u, v))d(y, v)}{1 + d(x, u) + d(y, v) + d(u, T(x, y))}
+ a_6 \frac{d(u, T(u, v))d(y, v)}{1 + d(x, u) + d(y, v) + d(u, T(x, y))}
+ a_7 \frac{d(u, T(x, y))d(x, u)}{1 + d(x, u) + d(y, v) + d(u, T(x, y))}
+ a_8 \frac{d(u, T(x, y))d(y, v)}{1 + d(x, u) + d(y, v) + d(u, T(x, y))}
+ a_9 \max\{d(u, T(x, y)), d(T(x, y), T(u, v))\}
\]

for all \(x, y, u, v \in X\) and \(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9 \geq 0\) with \(a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_9 < 1\) and \(a_1 + a_3 + a_4 + a_5 + a_7 + a_8 + a_9 < 1\). Then \(S\) and \(T\) have a unique common coupled fixed point.

Proof. The proof follows from Theorem 3.1 by taking \(S = T\).

Theorem 3.3. Let \((X, d)\) be a complete complex valued metric space, and let the mappings \(S, T: X \times X \to X\) satisfy

\[
d(S(x, y), T(u, v)) \leq a_1 \frac{d(x, u) + d(y, v)}{2} + a_2 \frac{d(x, S(x, y))d(S(x, y), T(u, v))}{1 + d(x, u) + d(y, v) + d(u, S(x, y)) + d(x, T(u, v))}
+ a_3 \max\{d(u, S(x, y)), d(S(x, y), T(u, v))\}
\]

for all \(x, y, u, v \in X\) and \(a_1, a_2, a_3 \geq 0\) with \(a_1 + a_2 + a_3 < 1\). Then \(S\) and \(T\) have a unique common coupled fixed point.

Proof. Take two arbitrary points \(x_0, y_0 \in X\). Define \(x_{2k+1} = S(x_{2k}, y_{2k}), y_{2k+1} = S(y_{2k}, x_{2k}), x_{2k+2} = T(x_{2k+1}, y_{2k+1})\) and \(y_{2k+2} = T(y_{2k+1}, x_{2k+1})\) for \(k = 0, 1, 2, \ldots\)
Adding (7) and (8), we get
\[ d(x_{2k+1}, x_{2k+2}) = d(S(x_{2k}, y_{2k}), T(x_{2k+1}, y_{2k+1})) \]
\begin{align*}
& \leq a_1 \frac{d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1})}{2} \\
& \quad + \frac{a_2}{2} d(x_{2k}, S(x_{2k}, y_{2k}))d(S(x_{2k}, y_{2k}), T(x_{2k+1}, y_{2k+1})) \\
& \quad + a_3 \max\{d(x_{2k+1}, S(x_{2k}, y_{2k})), d(S(x_{2k}, y_{2k}), T(x_{2k+1}, y_{2k+1}))\} \\
& = a_1 \frac{d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1})}{2} \\
& \quad + a_2 \frac{1}{2} d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1}) + d(x_{2k+1}, x_{2k+1}) + d(x_{2k}, x_{2k+2}) \\
& \quad + a_3 \max\{d(x_{2k+1}, x_{2k+1}), d(x_{2k}, x_{2k+2})\} \\
\end{align*}

Then, we can prove
\[ d(x_{2k+1}, x_{2k+2}) \leq \frac{a_1}{2(1 - a_2 - a_3)} |d(x_{2k}, x_{2k+1})| + \frac{a_1}{2(1 - a_2 - a_3)} |d(y_{2k}, y_{2k+1})| \]
\[ = k|d(y_{2k}, y_{2k+1})| + |d(x_{2k}, x_{2k+1})| \]
\begin{align*}
|d(x_{2k+1}, x_{2k+2})| & \leq a_1 \frac{|d(x_{2k}, x_{2k+1})| + |d(y_{2k}, y_{2k+1})|}{2} \\
& \quad + a_2 \frac{|d(x_{2k}, x_{2k+1})||d(x_{2k+1}, x_{2k+2})|}{|1 + d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1}) + d(x_{2k}, x_{2k+2})|} \\
& \quad + a_3 \max\{|d(x_{2k+1}, x_{2k+1}), d(x_{2k}, x_{2k+2})|\} \\
\end{align*}
Since \[ |1 + d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1}) + d(x_{2k}, x_{2k+2})| > |d(x_{2k}, x_{2k+1})| \], so we get
\[ |d(x_{2k+1}, x_{2k+2})| \leq \frac{a_1}{2(1 - a_2 - a_3)} |d(x_{2k}, x_{2k+1})| + \frac{a_1}{2(1 - a_2 - a_3)} |d(y_{2k}, y_{2k+1})| \]
\[ = k|d(y_{2k}, y_{2k+1})| + |d(x_{2k}, x_{2k+1})| \]
Similarly we can prove
\[ |d(y_{2k+1}, y_{2k+2})| \leq \frac{a_1}{2(1 - a_2 - a_3)} |d(y_{2k}, y_{2k+1})| + \frac{a_1}{2(1 - a_2 - a_3)} |d(x_{2k}, x_{2k+1})| \]
\[ = k|d(y_{2k}, y_{2k+1})| + |d(x_{2k}, x_{2k+1})| \]
Adding (7) and (8), we get
\[ |d(x_{2k+1}, x_{2k+2})| + |d(y_{2k+1}, y_{2k+2})| \leq \frac{a_1}{(1 - a_2 - a_3)} |d(y_{2k}, y_{2k+1})| + |d(x_{2k}, x_{2k+1})| \\
= k|d(y_{2k}, y_{2k+1})| + |d(x_{2k}, x_{2k+1})| \]
where \( k = \frac{a_1}{(1 - a_2 - a_3)} < 1 \). Also

\[
d(x_{2k+2}, x_{2k+3}) = d(S(x_{2k+1}, y_{2k+1}), T(x_{2k+2}, y_{2k+2})) \\
\leq a_1 \left[ d(x_{2k+1}, x_{2k+2}) + d(y_{2k+1}, y_{2k+2}) \right] \\
+ a_2 \left[ \frac{1}{2} d(x_{2k+1}, x_{2k+3}) + d(y_{2k+1}, y_{2k+2}) + d(x_{2k+2}, y_{2k+2}) \right] \\
+ a_3 \max \{d(x_{2k+2}, S(x_{2k+1}, y_{2k+1})), d(S(x_{2k+1}, y_{2k+1}), T(x_{2k+2}, y_{2k+2})) \}
\]

Continuing the same process, we get

\[
|d(x_{2k+2}, x_{2k+3})| \leq a_1 \left[ |d(x_{2k+1}, x_{2k+2})| + |d(y_{2k+1}, y_{2k+2})| \right] \\
+ a_2 \left[ \frac{1}{2} |d(x_{2k+1}, x_{2k+3})||d(x_{2k+2}, x_{2k+3})| \right] \\
+ a_3 d(x_{2k+2}, x_{2k+3})
\]

Since \( |1 + d(x_{2k+1}, x_{2k+2}) + d(y_{2k+1}, y_{2k+2}) + d(x_{2k+2}, x_{2k+3})| > |d(x_{2k+1}, x_{2k+2})| \), so we get

\[
|d(x_{2k+2}, x_{2k+3})| \leq \frac{a_1}{2(1 - a_2 - a_3)} |d(x_{2k+1}, x_{2k+2})| + \frac{a_1}{2(1 - a_2 - a_3)} |d(y_{2k+1}, y_{2k+2})|
\]

(9)

\[
|d(y_{2k+2}, y_{2k+3})| \leq \frac{a_1}{2(1 - a_2 - a_3)} |d(y_{2k+1}, y_{2k+2})| + \frac{a_1}{2(1 - a_2 - a_3)} |d(x_{2k+1}, x_{2k+2})|
\]

(10)

Adding (9) and (10), we get

\[
|d(x_{2k+2}, x_{2k+3})| + |d(y_{2k+2}, y_{2k+3})| \leq \frac{a_1}{(1 - a_2 - a_3)} \left[ |d(x_{2k+2}, x_{2k+3})| + |d(y_{2k+2}, y_{2k+3})| \right]
\]

\[
= k |d(x_{2k+2}, x_{2k+3})| + |d(y_{2k+2}, y_{2k+3})| \\
= k^2 |d(y_{2k+2}, y_{2k+3})| + |d(x_{2k+2}, x_{2k+3})|
\]

Continuing the same process, we get

\[
|d(x_n, x_{n+1})| + |d(y_n, y_{n+1})| \leq k |d(x_n, x_{n+1})| + |d(y_n, y_{n+1})| \\
\leq k^2 |d(x_{n-2}, x_{n-1})| + |d(y_{n-2}, y_{n-1})| \\
\leq \cdots \leq k^n |d(x_1, x_2)| + |d(y_1, y_2)|
\]
If \(|d(x_n, x_{n+1})| + |d(y_n, y_{n+1})| = \delta_n\). Then \(\delta_n \leq k\delta_{n-1} \leq k^2\delta_{n-2} \leq \cdots \leq k^n\delta_0\).

Without loss of generality, we take \(m > n\). Since \(0 \leq k < 1\), so we get

\[
|d(x_n, x_m)| + |d(y_n, y_m)| \leq |d(x_n, x_{n+1})| + |d(y_n, y_{n+1})|
+ |d(x_{n+1}, x_{n+2})| + |d(y_{n+1}, y_{n+2})| + \cdots
+ |d(x_{m-1}, x_m)| + |d(y_{m-1}, y_m)|
\leq [k^n\delta_0 + k^{n+1}\delta_0 + \cdots + k^{m-1}\delta_0]
\leq k^n[1 + k + k^2 + \cdots]\delta_0
= \frac{k^n}{1-k}\delta_0 \to 0 \text{ as } n \to \infty.
\]

This implies that \([x_n]\) and \([y_n]\) are Cauchy sequence in \(X\). Since \(X\) is complete, there exists \(x, y \in X\) such that \(x_n \to x\) and \(y_n \to y\) as \(n \to \infty\). We now show that \(x = S(x, y)\) and \(y = S(y, x)\).

We suppose on the contrary that \(x \neq S(x, y)\) and \(y \neq S(y, x)\) so that \(0 < d(x, S(x, y)) = l_1\) and \(0 < d(y, S(y, x)) = l_2\); we would then have

\[
l_1 = d(x, S(x, y)) \leq d(x, x_{2k+2}) + d(x_{2k+2}, S(x, y))
\leq d(x, x_{2k+2}) + d(S(x, y), T(x_{2k+1}, y_{2k+1}))
\leq d(x, x_{2k+2}) + a_1 \frac{d(x, x_{2k+1}) + d(y, y_{2k+1})}{2}
\]

\[
+ a_2 \frac{1 + d(x, x_{2k+1}) + d(y, y_{2k+1}) + d(x_{2k+1}, S(x, y)) + d(x, T(x_{2k+1}, y_{2k+1}))}{2}
+ a_3 \max\{d(x_{2k+1}, S(x, y)), d(S(x, y), T(x_{2k+1}, y_{2k+1}))\}
= d(x, x_{2k+2}) + a_1 \frac{d(x, x_{2k+1}) + d(y, y_{2k+1})}{2}
\]

\[
+ a_2 \frac{1 + d(x, x_{2k+1}) + d(y, y_{2k+1}) + d(x_{2k+1}, S(x, y)) + d(x, x_{2k+2})}{2}
+ a_3 \max\{d(x_{2k+1}, S(x, y)), d(S(x, y), x_{2k+2})\}
\]

which implies that

\[
|l_1| = |d(x, S(x, y))| \leq |d(x, x_{2k+2})| + a_1 \frac{|d(x, x_{2k+1})| + |d(y, y_{2k+1})|}{2}
+ a_2 \frac{|d(x, S(x, y))|d(S(x, y), x_{2k+2})}{|d(x, S(x, y))|d(S(x, y), x_{2k+2})}
+ a_3 \max\{|d(x_{2k+1}, S(x, y))|, |d(S(x, y), x_{2k+2})|\}
\]

Since \([x_n]\) and \([y_n]\) are convergent to \(x\) and \(y\), therefore by taking limit as \(k \to \infty\) we get

\[
|d(x, S(x, y))|(1 - a_2 - a_3) \leq 0
\]

Since \(a_1 + a_2 + a_3 < 1\), therefore,

\[
1 - a_2 - a_3 > 0.
\]
Hence

$$|l_1| = |d(x, S(x, y))| \leq 0.$$  

Which is contradiction, so $|d(x, S(x, y))| = 0 \Rightarrow x = S(x, y)$. Similarly we can prove that $y = S(y, x)$. Also we can prove that $x = T(x, y)$ and $y = T(y, x)$. Thus $(x, y)$ is a common coupled fixed point of $S$ and $T$. We now show that $S$ and $T$ have a unique common coupled fixed point. For this, assume that $(x_1, y_1) \in X \times X$ is a second common coupled fixed point of $S$ and $T$. Then

$$d(x, x_1) = d(S(x, y), T(x_1, y_1))$$
$$\leq a_1 \frac{d(x, x_1) + d(y, y_1)}{2} + a_2 \frac{d(x, S(x, y))d(S(x, y), T(x_1, y_1))}{2}$$
$$+ a_3 \max \{d(x_1, S(x, y)), d(S(x, y), T(x_1, y_1))\}$$
$$= a_1 \frac{d(x, x_1) + d(y, y_1)}{2}$$
$$+ a_2 \frac{d(x, x_1) + d(y, y_1) + d(x_1, x) + d(x_1, x_1)}{2}$$
$$+ a_3 \max \{d(x_1, x), d(x_1, x_1)\}$$

Thus

$$|d(x, x_1)| \leq a_1 \frac{|d(x, x_1)| + |d(y, y_1)|}{2} + a_3 \max \{|d(x, x_1)|, |d(y, y_1)|\}$$

Therefore,

$$|d(x, x_1)|(1 - a_1 a_2 - a_3) \leq \frac{a_1}{2} |d(y, y_1)|$$
$$|d(x, x_1)| \leq \frac{a_1}{2 - 2a_3 - a_1} |d(y, y_1)|.$$  \hspace{1cm} (11)

Similarly, we can prove that

$$|d(y, y_1)| \leq \frac{a_1}{2 - 2a_3 - a_1} |d(x, x_1)|.$$  \hspace{1cm} (12)

Adding (11) and (12), we get

$$|d(x, x_1)| + |d(y, y_1)| \leq \frac{a_1}{2 - 2a_3 - a_1} \left[|d(x, x_1)| + |d(y, y_1)|\right]$$

$$\left(1 - \frac{a_1}{2 - 2a_3 - a_1}\right)\left[|d(x, x_1)| + |d(y, y_1)|\right] \leq 0.$$  

which is a contradiction because $a_1 + a_2 + a_3 < 1$. Thus, we get $x_1 = x$ and $y_1 = y$, which proves the uniqueness of common coupled fixed point of $S$ and $T$. \hfill \Box
Corollary 3.4. Let \((X, d)\) be a complete complex valued metric space, and let the mappings \(S: X \times X \to X\) satisfy
\[
d(S(x, y), S(u, v)) \leq a_1 \frac{d(x, u) + d(y, v)}{2} + a_2 \frac{d(x, S(x, y))d(S(x, y), S(u, v))}{1 + d(x, u) + d(y, v) + d(u, S(x, y)) + d(x, S(u, v))}
\]
for all \(x, y, u, v \in X\) and \(a_1, a_2, a_3 \geq 0\) with \(a_1 + a_2 + a_3 < 1\). Then \(S\) has a unique common coupled fixed point.

Proof. The proof follows from Theorem 3.3 by taking \(T = S\).

Example 3.5. Suppose \(X = [0, 1]\). Defined the function \(d: X \times X \to \mathbb{C}\) by \(d(x, y) = i|x - y|, \forall x, y \in X\). Clearly \((X, d)\) is complex valued metric space. If we define two mappings \(S, T: X \times X \to X\), as \(S(x, y) = \frac{x + y}{4}\), \(T(x, y) = \frac{x + y}{3}\) for each \(x, y \in X\). Then it can be proved simply that the maps \(S\) and \(T\) satisfy the condition of Theorem 3.1 with \(a_1 = \frac{1}{7}, a_2 = \frac{1}{6}, a_3 = \frac{1}{14}, a_4 = \frac{1}{15}, a_5 = \frac{1}{17}, a_6 = \frac{1}{16}, a_7 = \frac{1}{18}, a_8 = \frac{1}{19}, a_9 = \frac{1}{26}\). Hence \((0, 0)\) is a unique common coupled fixed point of \(S\) and \(T\).

References


