Square Divisor Cordial, Cube Divisor Cordial and Vertex Odd Divisor Cordial Labeling of Graphs

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Abstract

In this paper we investigate square divisor cordial labeling, cube divisor cordial labeling and vertex odd divisor cordial labeling of $K_{1,1,n}$, $K_2 + mK_1$, Umbrella, $C_n(t)$, $G =< K_1^{(1)}, K_1^{(2)} >$, arbitrary supersubdivision of $K_1,n$, the graph obtained by duplication of an edge in $K_{1,n}$ and $K_2,n \circ K_1$.

Key words: Square divisor cordial labeling, Cube divisor cordial labeling, Vertex odd divisor cordial labeling.

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1 Introduction

Here we consider simple, finite, connected and undirected graph $G = (V, E)$ with $p$ vertices and $q$ edges. For standard terminology and notations related to graph theory we refer to Gross and Yellen[2]. The most recent findings on various graph labeling techniques can be found in Gallian[1]. The brief summary of definitions and other information which are necessary for the present investigation are provided below.

1.1 Definitions

Definition 1.1.1. [8] Let $G = (V, E)$ be a graph. A bijection $f : V(G) \rightarrow \{1,2,\ldots,|V(G)|\}$ is said to be divisor cordial labeling of graph $G$ if the induced function $f^* : E(G) \rightarrow \{0,1\}$ defined by

$$f^*(e = uv) = \begin{cases} 1; & \text{if } f(u)f(v) \text{ or } f(v)f(u). \\ 0; & \text{otherwise.} \end{cases}$$

satisfies the condition $|e_f(0) - e_f(1)| \leq 1.$

A graph which admits divisor cordial labeling is called a divisor cordial graph.

Definition 1.1.2. [4] Let $G = (V, E)$ be a graph, $f : V(G) \rightarrow \{1,2,\ldots,|V(G)|\}$ be a bijection and the induced function $f^* : E(G) \rightarrow \{0,1\}$ is defined by

$$f^*(e = uv) = \begin{cases} 1; & \text{if } |f(u)|^2|f(v)| + |f(v)|^2|f(u)|. \\ 0; & \text{otherwise.} \end{cases}$$

Then function $f$ is called square divisor cordial labeling if $|e_f(0) - e_f(1)| \leq 1.$

A graph which admits square divisor cordial labeling is called square divisor cordial graph.

Definition 1.1.3. [3] Let $G = (V, E)$ be a graph, $f : V(G) \rightarrow \{1,2,\ldots,|V(G)|\}$ be a bijection and the induced function $f^* : E(G) \rightarrow \{0,1\}$ is defined by

$$f^*(e = uv) = \begin{cases} 1; & \text{if } f(u)f(v) \text{ or } f(v)f(u). \\ 0; & \text{otherwise.} \end{cases}$$

Then function $f$ is called cube divisor cordial labeling if $|e_f(0) - e_f(1)| \leq 1.$

A graph which admits cube divisor cordial labeling is called cube divisor cordial graph.

Definition 1.1.4. [5] Let $G = (V, E)$ be a graph, $f : V(G) \rightarrow \{1,3,\ldots,2n-1\}$ be a bijection and the induced function $f^* : E(G) \rightarrow \{0,1\}$ is defined by

$$f^*(e = uv) = \begin{cases} 1; & \text{if } f(u)f(v) \text{ or } f(v)f(u). \\ 0; & \text{otherwise.} \end{cases}$$

Then function $f$ is called vertex odd divisor cordial labeling if $|e_f(0) - e_f(1)| \leq 1.$

A graph which admits vertex odd divisor cordial labeling is called vertex odd divisor cordial graph.
Definition 1.1.5. [1] A graph is called a tripartite graph if we can divide the vertex set of the graph into three disjoint non empty subsets $V_1$, $V_2$ and $V_3$ so that vertices in the same set are not adjacent to each other. The complete tripartite graph with $|V_1| = n_1$, $|V_2| = n_2$, $|V_3| = n_3$ is denoted by $K_{n_1,n_2,n_3}$.

Definition 1.1.6. [1] Umbrella is the graph obtained from fan by joining a path $P_m$ to a middle vertex of path $P_n$. It is denoted by $U(m,n)$.

Definition 1.1.7. [1] The joinsum of complete bipartite graphs $<K_{1,n}^{(1)}, K_{1,n}^{(2)}, \ldots, K_{1,n}^{(t)}>\, is\, the\, graph\, obtained\, by\, starting\, with\, t\, copies\, of\, K_{1,n}$ namely $K_1, K_{1,n}, \ldots, K_{1,n}^{(t)}$ and joining apex vertex of each pair $K_{1,n}^{(i)}$ and $K_{1,n}^{(i+1)}$ to a new vertex $v_i$ where $1 \leq i \leq t - 1$.

Definition 1.1.8. [1] Let $G$ be a graph with $n$ vertices and $e$ edges. A graph $H$ is said to be a supersubdivision of $G$ if $H$ is obtained from $G$ by replacing every edge $e_i$ of $G$ by complete bipartite graph $K_{2,m_i},$ for some $m_i$, $1 \leq i \leq n$ in such a way that the ends of $e_i$ are merged with the two vertices of 2-vertices part of $K_{2,m_i}$ after removing the edge $e_i$ from $G$.

Definition 1.1.9. [1] Duplication of an edge $e = uv$ of a graph $G$ produces a new graph $G'$ by adding an edge $e' = u'v'$ such that $N(u') = N(u) \cup \{v\}$ and $N(v') = N(v) \cup \{u\}$.

Definition 1.1.10. [9] Let $(V_1, V_2)$ be the bipartition of $K_{m,n}$, where $V_1 = \{u_1, u_2, \ldots, u_m\}$ and $V_2 = \{v_1, v_2, \ldots, v_n\}$. The graph $K_{m,n} \cup u_i(K_1)$ is defined by attaching a pendant vertex to the vertex $u_i$ for some $i$.

Notation 1.1. [1] The one point union of $t \geq 1$ cycles, each of length $n$ is denoted by $C_n^t$.

2 Main Results

Theorem 2.1 $K_{1,1,n}$ is square divisor cordial graph for each $n$.

Proof: Let $u$ and $v$ be the vertices of degree $n$ and $u_1, u_2, \ldots, u_n$ be the vertices of degree 2, $|V(K_{1,1,n})| = n + 2$, $|E(K_{1,1,n})| = 2n + 1$. We define vertex labeling $f : V(K_{1,1,n}) \to \{1, 2, 3, \ldots, n + 2\}$ as follows $f(u) = 1$, $f(v) = p$, where $p$ is largest prime number.

The remaining vertices $u_1, u_2, \ldots, u_n$ of $K_{1,1,n}$ can be labeled by remaining labels $2, 3, \ldots, p - 1, p + 1, \ldots, n$ in any order.

Then we get $|e_f(1)| = e_f(0) + 1$.

Thus $K_{1,1,n}$ is a square divisor cordial graph.

Illustration 2.1 Square divisor cordial labeling of graph $K_{1,1,6}$ is shown in Fig. 1 as an illustration for the Theorem 2.1.

Corollary 2.1. $K_{1,1,n}$ is a cube divisor cordial graph.

Proof. The vertex labeling function can be defined same as in Theorem 2.1. One can observe that the vertex labeling function satisfies condition for cube divisor cordial labeling. Hence $K_{1,1,n}$ is a cube divisor cordial graph.

Corollary 2.2. $K_{1,1,n}$ is a vertex odd divisor cordial graph.

Proof. The vertex labeling function can be defined similar as in Theorem 2.1. One can observe that the vertex labeling function satisfies condition for vertex odd divisor cordial labeling. Hence $K_{1,1,n}$ is a vertex odd divisor cordial graph.

Theorem 2.2 $K_2 + mK_1$ is a square divisor cordial graph.

Proof: Let $u$ and $v$ be the vertices of degree $m + 1$ and $u_1, u_2, \ldots, u_m$ be the vertices of degree 2 in $K_2 + mK_1$. Here $|V(K_2 + mK_1)| = m + 2$ and $|E(K_2 + mK_1)| = 2m + 1$.

We define vertex labeling $f : V(K_2 + mK_1) \to \{1, 2, 3, \ldots, m + 2\}$ as follows $f(u) = 1$, $f(v) = p$, where $p$ is largest prime number and label the remaining labels to the remaining vertices $u_1, u_2, \ldots, u_m$ in any order.

Here $|e_f(1)| = m + 1$ and $|e_f(0)| = m$.

Clearly it satisfies the condition $|e_f(1)| = e_f(0) \leq 1$.

Thus $K_2 + mK_1$ is a square divisor cordial graph.

Illustration 2.2 Square divisor cordial labeling of the graph $G = K_2 + 7K_1$ is shown in Fig. 2 as an illustration for the Theorem 2.2.

Corollary 2.3. $K_2 + mK_1$ is a cube divisor cordial graph.
Corollary 2.4. $K_2 + mK_1$ is a vertex odd divisor cordial graph.

Theorem 2.3 Umbrella $U(n, 3)$ is a square divisor cordial graph.
Proof: Let $u$ be the vertex of degree $n$, $v$ be the vertex of degree 2, $w$ be the pendant vertex and $u_1, u_2, \ldots, u_n$ be the vertices of path $P_n$. Here $|V(U(n, 3))| = n+3$ and $|E(U(n, 3))| = 2n+1$. We define vertex labeling $f : V(U(n, 3)) \rightarrow \{1, 2, 3, \ldots, n+3\}$ as per the following cases.
Case 1: $n$ is odd.
$f(u) = 1$, $f(u_i) = i+1$, $1 \leq i \leq n$.
$f(v) = n+2$, $f(w) = n+3$.
Here $e_f(1) = e_f(0) - 1$.
Case 2: $n$ is even.
$f(u) = 1$, $f(u_i) = i+1$, $1 \leq i \leq n$.
$f(v) = n+2$, $f(w) = n+3$.
Here $e_f(1) = e_f(0) + 1$.
In each case we have $|e_f(0) - e_f(1)| \leq 1$. Hence Umbrella $U(n, 3)$ is a square divisor cordial graph.

Illustration 2.3 Square divisor cordial labeling of the graph $U(5,3)$ and $U(6,3)$ is shown in Fig. 3 and Fig. 4 respectively as an illustration for Theorem 2.3.

Corollary 2.5. Umbrella $U(n, 3)$ is a cube divisor cordial graph.

Corollary 2.6. Umbrella $U(n, 3)$ is a vertex odd divisor cordial graph.

Theorem 2.4 $C_4^{(t)}$ is a square divisor cordial graph.
Proof: Let $v_1^{(1)}, v_2^{(1)}, v_3^{(1)}, v_4^{(1)}, 1 \leq i \leq t$ be the vertices of $C_4^{(t)}$ with $v_1^{(1)} = v_2^{(1)} = v_3^{(1)} = \ldots = v_4^{(1)} = v$.
$|V(C_4^{(t)})| = 3t+1$, $|E(C_4^{(t)})| = 4t$.
We define vertex labeling $f : V(C_4^{(t)}) \rightarrow \{1, 2, 3, \ldots, 3t+1\}$ as follows.
$f(v) = 1$, $f(v_2^{(1)}) = 3t - 1$, $f(v_3^{(1)}) = 3t$.
$f(v_4^{(1)}) = 3t + 1$, $1 \leq i \leq t$.
Here $e_f(1) = e_f(0) = 2t$.
Thus $C_4^{(t)}$ is a square divisor cordial graph.

Illustration 2.4 Square divisor cordial labeling of the graph $C_4^{(4)}$ is shown in Fig. 5 as an illustration for Theorem 2.4.

Figure 2: Square divisor cordial labeling of the graph $G = K_2 + 7K_1$

Figure 3: Square divisor cordial labeling of the graph $U(5,3)$

Figure 4: Square divisor cordial labeling of the graph $U(6,3)$

Figure 5: Square divisor cordial labeling of the graph $C_4^{(4)}$

Corollary 2.7. $C_4^{(t)}$ is a cube divisor cordial graph.

Corollary 2.8. $C_4^{(t)}$ is a vertex odd divisor cordial graph.

Theorem 2.5 The graph $< K_{1,n}^{(1)}, K_{1,n}^{(2)} >$ is a square divisor cordial graph.
Proof: Let $G = < K_{1,n}^{(1)}, K_{1,n}^{(2)} >$.
Let $v_1^{(1)}, v_2^{(1)}, v_3^{(1)}, \ldots, v_n^{(1)}$ be the pendant vertices of $K_{1,n}^{(1)}$ and $v_1^{(2)}, v_2^{(2)}, v_3^{(2)}, \ldots, v_n^{(2)}$ be the pendant vertices of $K_{1,n}^{(2)}$.
of $K_{1,n}^{(2)}$. Let $v_0^{(1)}$ and $v_0^{(2)}$ be the apex vertices of $K_{1,n}^{(1)}$ and $K_{1,n}^{(2)}$, respectively which are adjacent to a new common vertex say $x$.

Here $|V(G)| = 2n + 3$, $|E(G)| = 2n + 2$.

Now assign label 1 to $v_0^{(1)}$ and label $v_0^{(2)}$ by the largest prime number $p$ such that $p \leq 2n + 3$. Label the vertices $v_1^{(1)}, v_2^{(1)}, \ldots, v_n^{(1)}, v_1^{(2)}, v_2^{(2)}, \ldots, v_n^{(2)}, x$ by the remaining labels in any order. Then we get $|e_f(1)| = |e_f(0)| = n + 1$.

Thus $< K_{1,n}^{(1)}, K_{1,n}^{(2)} >$ is a square divisor cordial graph.

**Illustration 2.5** Square divisor cordial labeling of the graph $G = < K_{1,3}^{(1)}, K_{1,3}^{(2)} >$ is shown in Fig. 6 as an illustration for Theorem 2.5.

![Figure 6: Square divisor cordial labeling of the graph $G = < K_{1,3}^{(1)}, K_{1,3}^{(2)} >$](image)

**Corollary 2.9.** The graph $< K_{1,n}^{(1)}, K_{1,n}^{(2)} >$ is a cube divisor cordial graph.

**Corollary 2.10.** The graph $< K_{1,n}^{(1)}, K_{1,n}^{(2)} >$ is a vertex odd divisor cordial graph.

**Theorem 2.6** The graph Arbitrary supersubdivision of $K_{1,n}$ is a square divisor cordial graph.

**Proof:** Let $v_0, v_1, v_2, \ldots, v_n$ be the vertices of $K_{1,n}$ and let $e_i$ denote the edge $v_iv_i$ of $K_{1,n}$ for $1 \leq i \leq n$.

Let $G$ be the graph obtained by arbitrary supersubdivision of $K_{1,n}$ in which each edge $e_i$ of $K_{1,n}$ is replaced by a complete bipartite graph $K_{2,m_i}$ and $u_{ij}$ be the vertices of $m_i$-vertices part, $1 \leq i \leq n, 1 \leq j \leq m_i$.

Observe that $G$ has $2(m_1, m_2, \ldots, m_n)$ edges.

We define vertex labeling $f : V(G) \rightarrow \{1, 2, 3, \ldots, n + 2\}$ as $f(v_0) = 1$ and label the vertices $v_i, 1 \leq i \leq n$ by the last $n$ consecutive prime numbers. Assign remaining labels to the remaining vertices $u_{ij}, 1 \leq i \leq n, 1 \leq j \leq m_i$ in any order. Here $e_f(1) = e_f(0)$. Thus arbitrary supersubdivision of $K_{1,n}$ is a square divisor cordial graph.

**Illustration 2.6** Square divisor cordial labeling of arbitrary supersubdivision of $K_{1,4}$ is shown in Fig. 7 as an illustration for Theorem 2.6.

**Remark 2.1.** Arbitrary supersubdivision of $K_{1,n}$ is nothing but the one vertex union of the complete bipartite graphs $K_{2,m_i}$, where $v_i$ is arbitrary, $1 \leq i \leq n$.

**Corollary 2.11.** Arbitrary supersubdivision of $K_{1,n}$ is a cube divisor cordial graph.

**Corollary 2.12.** Arbitrary supersubdivision of $K_{1,n}$ is a vertex odd divisor cordial graph.

**Theorem 2.7** The graph obtained by duplication of an edge in $K_{1,n}$ is a square divisor cordial graph.

**Proof:** Let $v_0$ be the apex vertex and $v_1, v_2, \ldots, v_n$ be the successive pendant vertices of $K_{1,n}$.

Let $G$ be the graph obtained by duplication of an edge $e = v_0v_n$ by a new edge $e' = v_0'v_n'$.

Hence in $G$, $\deg(v_0) = n, \deg(v_0') = n, \deg(v_{n}) = 1$, $\deg(v_{n}') = 1$ and $\deg(v_i) = 2; 1 \leq i \leq n - 1$.

$|V(G)| = n + 3, |E(G)| = 2n$.

We define vertex labeling $f : V(G) \rightarrow \{1, 2, 3, \ldots, n + 3\}$ as follows.

$f(v_0) = 0, f(v_0') = p$, where $p$ is the largest prime number. $f(v_{n}) = n + 2, f(v_{n}') = n + 3, f(v_i) = i + 1; 1 \leq i \leq n - 1$.

Here $e_f(1) = e_f(0) = n$.

Thus the graph obtained by duplication of an edge in $K_{1,n}$ is a square divisor cordial graph.

**Illustration 2.7** Square divisor cordial labeling of the graph obtained by duplication of an edge in $K_{1,8}$ is shown in Fig. 8 as an illustration for Theorem 2.7.

**Corollary 2.13.** The graph obtained by duplication of an edge in $K_{1,n}$ is a cube divisor cordial graph.

**Corollary 2.14.** The graph obtained by duplication of an edge in $K_{1,n}$ is a vertex odd divisor cordial graph.

**Theorem 2.8** $K_{2,n} \oplus u_2(K_1)$ is a square divisor cordial graph.

**Proof:** Let $G = K_{2,n} \oplus u_2(K_1)$. Let $V = V_1 \cup V_2$ be the bipartition of $K_{2,n}$ such that $V_1 = \{v_1, v_2\}$ and $V_2 = \{v_1, v_2, \ldots, v_n, w\}$.

![Figure 7: Square divisor cordial labeling of arbitrary supersubdivision of $K_{1,4}$](image)
Let \( w \) be the pendant vertex adjacent to \( u_2 \) in \( G \).

\[ |V(G)| = n + 3, |E(G)| = 2n + 1. \]

We define vertex labeling

\[ f : V(G) \to \{1, 2, 3, \ldots, n + 3\} \]

as follows.

\[ f(u_1) = 1, f(u_2) = p, \]

where \( p \) is the largest prime number. Label the remaining labels to the remaining vertices \( v_1, v_2, \ldots, v_n \) in any order.

Here \( e_f(1) = n, e_f(0) = n + 1 \).

Thus \( K_{2,n} \odot u_2(K_1) \) is a square divisor cordial graph.

**Illustration 2.8** Square divisor cordial labeling of the graph \( G = K_{2,5} \odot u_2(K_1) \) is shown in Fig. 9 as an illustration for **Theorem 2.8**.

To discuss the natural relation between these labelings, if any. The discussion and further scope of research in this area are left to the reader.

### References


