Some Common Fixed Point Theorem in Fuzzy Metric Space

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Abstract - In this paper we are generalizing the results of Som [3,4], Mukherjee [2] and Shrivastava[1].

Keywords - semi-compatible and (B, T) is weak-compatible, Fuzzy Metric Space.

MAIN RESULT

Theorem 1. Let A, B, S and T be self mappings of a complete fuzzy metric space \((X, M, *)\) satisfying

(a) \(A(X) \subseteq T(X), B(X) \subseteq S(X),\)

(b) one of A or S is continuous,

(c) the pair \((A, S)\) is semi-compatible and \((B, T)\) is weak-compatible,

(d) \(aM(Ax, By, t) - bM(Sx, Ty, t) > \Phi\{M(Sx, Ty, t), M(Sx, Ax, t), M(Sx, By, t), M(Ty, Ax, t), M(Ty, By, t)\},\)

where \(\Phi : (R^+)^5 \rightarrow R^+\) is continuous and strictly increasing in each co-ordinate variable such that for all \(x, y \in X, a < b + 1\) and for any \(\nu < 1,\)

\(\Phi (\nu, \nu, a, a, \nu, a) > \nu, a_1 + a_2 = 3.\) Then A, B, S and T have a unique common fixed point in X.

Now, we are proving this theorem of shrivastava[1] for six mapping

Theorem 2. Let, A, B, S, T, P and Q be self mappings of a complete fuzzy metric space \((X, M, *)\) satisfying

(a) \(A(X) \subseteq PT(X), B(X) \subseteq QS(X),\)

(b) one of A or QS is continuous,

(c) the pair \((A, QS)\) is semi compatible and \((B, PT)\) is weak compatible,

(d) \(aM(Ax, By, t) - bM(QSx, PTy, t) > \Phi\{M(QSx, PTy, t), M(QSx, Ax, t), M(QSx, By, t), M(PTx, Ax, t), M(PTy, By, t)\},\)

where \(\Phi\) is a continuous mapping \(\Phi : (R^+)^5 \rightarrow R\) satisfying \(\nu < 1,\)

\(\Phi (\nu, \nu, a, a, \nu, a) > \nu, a_1 + a_2 = 3.\) Then A, B, S, T, P and Q have a unique common fixed point in X.
Proof. Proof of theorem 1, let $x_0$ be any arbitrary point. Since $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$ then there exists $x_1, x_2 \in X$ such that

$$A x_0 = T x_1 = y_1, \quad B x_1 = S x_2 = y_2.$$ 

Inductively, construct two sequences $\{y_n\}$ and $\{x_n\}$ in $X$ such that

$$y_{2n+1} = A x_{2n} = T x_{2n+1},$$

$$y_{2n+2} = B x_{2n+1} = S x_{2n+2}; \quad n = 0, 1, 2, 3, \ldots$$

Let $M_n = M(y_n, y_{n+1}, t); \quad n = 0, 1, 2, 3, \ldots$

We claim that $\{M_n\}$ is a increasing sequence, suppose on the contrary that

$$M_{2n} > M_{2n+1}, \text{ for some } n.$$ 

Putting $x = x_{2n}$ and $y = x_{2n+1}$ in (d), we get

$$a M(A x_{2n}, B x_{2n+1}, t) - b M(S x_{2n}, T x_{2n+1}, t) \geq \phi \{M(S x_{2n}, T x_{2n+1}, t), M(S x_{2n}, A x_{2n}, t), M(T x_{2n+1}, A x_{2n}, t), M(T x_{2n+1}, A x_{2n+1}, t)\}.$$ 

$$\Rightarrow a M(y_{2n+1}, y_{2n+2}, t) - b M(y_{2n}, y_{2n+1}, t),$$ 

$$\geq \phi \{M(y_{2n}, y_{2n+1}, t), M(y_{2n}, Y_{2n+1}, t), M(y_{2n}, Y_{2n+2}, t), M(y_{2n+1}, Y_{2n+1}, t), M(y_{2n+1}, Y_{2n+2})\}$$

$$\Rightarrow a M_{2n+1} - b M_{2n} \geq \phi \{M_{2n}, M_{2n}, M_{2n} + M_{2n+1}, 1, M_{2n+1}\}$$

$$\Rightarrow \frac{b}{a-1} M_{2n} > M_{2n+1}$$

$$\Rightarrow M_{2n+1} > M_{2n} \quad [\Theta a < b + 1]$$

which is a contradiction.

Thus $\{M_n\}$ is increasing sequence of positive real number in $[0, 1]$ and therefore $\lim_{n \to \infty} M_n = 1$. Now, we show that

$\{y_n\}$ is a cauchy sequence. Since $\lim_{n \to \infty} M_n = 1$, it is sufficient to show that $\{y_{2n}\}$ is a cauchy sequence.

Suppose that it is not so, then there is an $\varepsilon > 0$ such that for each integer $2k$
There exists even integer $2nk$ and $2mk$ with $2k < 2nk < 2mk$ such that

$$M(y_{2nk}, y_{2mk}, t) \leq 1 - \varepsilon; \text{ for some } t > 0. \quad (1)$$

Let for each even integer $2k, 2mk$ be the least positive integer exceeding $2nk$ satisfying (1),

then

$$M(y_{2nk}, y_{2mk-2}, t) > 1 - \varepsilon \quad \text{and}$$

$$M(y_{2nk}, y_{2mk}, t) \leq 1 - \varepsilon. \quad (2)$$

As such, for each even integer $2k$, we have

$$1 - \varepsilon > M(y_{2nk}, y_{2mk}, t) \geq M(y_{2nk}, y_{2mk-2}, t/3) \times M(y_{2mk-2}, y_{2mk-1}, t/3) \times M(y_{2mk-1}, y_{2mk}, t/3).$$

so by (2) and as $k \to \infty$, we get

$$\lim_{k \to \infty} M(y_{2nk}, y_{2mk}, t) = 1 - \varepsilon. \quad (3)$$

Now, using (3) in the triangular inequalities

$$M(y_{2nk}, y_{2mk-1}, t) \geq M(y_{2nk}, y_{2mk}, t/2) \times M(y_{2mk}, y_{2mk-1}, t/2)$$

and

$$M(y_{2nk+1}, y_{2mk-1}, t) \geq M(y_{2nk+1}, y_{2nk}, t/3) \times M(y_{2nk}, y_{2mk}, t/3) \times M(y_{2mk}, y_{2mk-1}, t/3).$$

Taking $k \to \infty$, then

$$M(y_{2nk+1}, y_{2mk-1}, t) \geq 1 - \varepsilon \times 1 = 1 - \varepsilon$$

and

$$M(y_{2nk+1}, y_{2mk-1}, t) \geq 1 \times 1 - \varepsilon \times 1 = 1 - \varepsilon.$$
\[ \geq M(y_{2nk}, y_{2nk+1}, t/2) \ast \frac{1}{a} \phi \{ M(y_{2mk-1}, y_{2nk}, t/2), M(y_{2mk-1}, y_{2mk}, t/2), M(y_{2mk-1}, y_{2mk+1}, t/2), \]
\[ M(y_{2mk}, y_{2nk}, t/2), \{ M(y_{2nk}, y_{2nk+1}) \} + \frac{b}{a} M(y_{2mk-1}, y_{2nk}, t/2). \]

On taking \( k \to \infty \)

\[ 1 - \varepsilon \geq \frac{1}{a} \phi \{ 1 - \varepsilon, 0, 1 - \varepsilon, 1 - \varepsilon, 0 \} + \frac{b}{a} (1 - \varepsilon) \]
\[ > \frac{1}{a} (1 - \varepsilon) + \frac{b}{a} (1 - \varepsilon) = \frac{1 + b}{a} (1 - \varepsilon) \]
\[ \Rightarrow 1 - \varepsilon > 1 - \varepsilon \]

which is a contradiction.

Hence \( \{ y_{2n} \} \) is a cauchy sequence in \( X \). By completeness of \( X \), \( \{ y_n \} \) converges to \( z \in X \). Hence, the subsequences

\[ \{ Ax_{2n} \} \to z, \quad \{ Sx_{2n} \} \to z, \quad \{ Tx_{2n+1} \} \to z, \quad \{ Bx_{2n+1} \} \to z. \quad (4) \]

\[ \text{and} \]
\[ \{ Ax_{2n} \} \to z, \quad \{ Sx_{2n} \} \to z, \quad \{ Bx_{2n+1} \} \to z. \quad (5) \]

Since the limit of a sequence in fuzzy metric space is unique we obtain that

\[ Az = Sz \]

**Step 1.** Now, we will prove that \( Az = z \). Suppose on the contrary \( Az \neq z \).

By putting \( x = z, y = x_{2n+1} \) in \( (d) \) we have

\[ aM(Az, Bx_{2n+1}, t) - bM(Sz, Tx_{2n+1}, t) \]
\[ \geq \phi \{ M(Sz, Tx_{2n+1}, t), M(Sz, Az, t), M(Sz, Bx_{2n+1}, t), M(Tx_{2n+1}, Az, t), M(Tx_{2n+1}, Bx_{2n+1}, t) \} \]
\[ \Rightarrow aM(Az, z, t) - bM(Az, z, t) \]
\[ \geq \phi \{ M(Az, z, t), M(Az, Az, t), M(Az, z, t), M(z, Az, t), M(z, z, t) \} \]
\[ \geq \phi \{ M(Az, z, t), 1, M(Az, z, t), M(Az, z, t), 1 \} \]
\[ \geq \phi \{ M(Az, z, t), M(Az, z, t), 2M(Az, z, t), M(Az, z, t), M(Az, z, t) \} \]

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\[ (a - b) \ M(Az, z, t) > M(Az, z, t) \]

which is a contradiction.

Hence \( z = Az = Sz \).

**Step 2.** Since \( A(X) \subseteq T(X) \), there exists \( u \in X \) such that

\[ z = Az = Tu. \]

Now, we have to prove that \( z = Bu \), suppose on the contrary that \( z \neq Bu \)

Putting \( x = x_{2n}, y = u \) in (d) we get.

\[ aM(Ax_{2n}, Bu, t) - bM(Sx_{2n}, Tu, t) \]

\[ \geq \ \emptyset \{ M(Sx_{2n}, Tu, t), M(Sx_{2n}, Ax_{2n}, t), M(Sx_{2n}, Bu, t), M(Tu, Ax_{2n}, t), M(Tu, Bu, t) \}. \]

On taking limit as \( n \to \infty \) and using (4) we obtain that

\[ aM(z, Bu, t) - bM(z, z, t) \geq \emptyset \{ M(z, z, t), M(z, Bu, t), M(z, Bu, t) \} \]

\[ \Rightarrow \ aM(z, Bu, t) - bM(z, z, t) \geq \emptyset \{ 1, 1, M(z, Bu, t) \} \]

\[ aM(z, Bu, t) - bM(z, Bu, t) \geq \emptyset \{ M(z, Bu, t), M(z, Bu, t), 2M(z, Bu, t), M(z, Bu, t), M(z, Bu, t) \} \]

\[ (a - b) \ M(z, Bu, t) > M(z, Bu, t) \]

which is a contradiction.

Hence \( z = Bu = Tu \) and the weak compatibility of \( (B, T) \) gives

\[ TBu = BTu \]

i.e. \( Tz = Bz \)

**Step 3.** By putting \( x = z, \ y = z \) in (d) and assuming \( Az \neq Bz \), we have.

\[ aM(Az, Bz, t) - bM(Sz, Tz, t) \]

\[ \geq \emptyset \{ M(Sz, Tz, t), M(Sz, Az, t), M(Sz, Bz, t), M(Tz, Az, t), M(Tz, Bz, t) \} \]

\[ \Rightarrow \ aM(Az, Bz, t) - bM(Az, Bz, t) \]

\[ \geq \emptyset \{ M(Az, Bz, t), M(Az, Az, t), M(Az, Bz, t), M(Bz, Az, t), M(Bz, Bz, t), M(Tz, Tz, t) \} \]
\[(a-b) \, M(Az, Bz, t) \geq \varphi \{ M(Az, Bz, t), 1, M(Az, Bz, t), M(Az, Bz, t), 1 \} \]

\[\geq \varphi \{ M(Az, Bz, t), M(Az, Bz, t), 2M(Az, Bz, t), M(Az, Bz, t), M(Az, Bz, t) \} \]

\[(a-b) \, M(Az, Bz, t) > M(Az, Bz, t) \]

which is a contradiction. Hence \(Az = Bz\).

Combining the result from **Steps 1, 2, 3** we obtain that

\[z = Az = Bz = Sz = Tz\]

Therefore \(z\) is a common fixed point of \(A, B, S\) and \(T\).

**Case 2.** \(S\) is continuous

As \(S\) is continuous and \((A, S)\) is semi-compatible, we have.

\[SAx_{2n} \rightarrow Sz, \quad S^2x_{2n} \rightarrow Sz, \quad ASx_{2n} \rightarrow Sz \quad (6)\]

Thus,

\[\lim_{n \to \infty} SAx_{2n} = \lim_{n \to \infty} ASx_{2n} = Sz\]

we prove \(Sz = z\), suppose on the contrary that \(Sz \neq z\).

**Step 4.** Putting \(x = Sx_{2n}, y = x_{2n+1}\) in (d)

\[aM(ASx_{2n}, Bx_{2n+1}, t) - bM(SSx_{2n}, Tx_{2n+1}, t) \]

\[\geq \varphi \{ M(SSx_{2n}, Tx_{2n+1}, t), M(SSx_{2n}, ASx_{2n}, t), M(SSx_{2n}, Bx_{2n+1}, t), M(Tx_{2n+1}, ASx_{2n}, t), M(Tx_{2n+1}, Bx_{2n+1}, t) \} \]

\[\Rightarrow aM(Sz, z, t) - bM(Sz, z, t) \]

\[\geq \varphi \{ M(Sz, z, t), M(Sz, Sz, t), M(Sz, z, t), M(z, Sz, t), M(z, z, t) \} \]

\[\geq \varphi \{ M(Sz, z, t), 1, M(Sz, z, t), M(Sz, z, t), 1 \} \]

\[\geq \varphi \{ M(Sz, z, t), M(Sz, z, t), 2M(Sz, z, t), M(Sz, z, t), M(Sz, z, t) \} \]

\[\Rightarrow (a-b) \, M(Sz, z, t) > M(Sz, z, t) \]

which is a contradiction. Hence \(Sz = z\).

**Step 5.** By putting \(x = z, y = x_{2n+1}\) in (d)
aM(Az, Bx_{2n+1}, t) - bM(Sz, Tx_{2n+1}, t)

\geq \big\{M(Sz, Tx_{2n+1}, t), M(Sz, Az, t), M(Sz, Bx_{2n+1}, t), M(Tx_{2n+1}, Az, t), M(Tx_{2n+1}, Bx_{2n+1}, t)\big\}

\Rightarrow aM(Az, z, t) - bM(z, z, t) \geq \big\{M(z, z, t), M(z, Az, t), M(z, z, t), M(z, Az, t), M(z, z, t)\big\}

\Rightarrow aM(Az, z, t) - b > \big\{1, M(Az, z, t), 1, M(Az, z, t)\big\}

\Rightarrow aM(Az, z, t) - b (Az, z, t) > \big\{M(Az, z, t), M(Az, z, t), 2M(Az, z, t), M(Az, z, t), M(Az, z, t)\big\}

\Rightarrow (a-b) M(Az, z, t) > M(Az, z, t)

Which is a contradiction.

Hence Az = z = Sz.

Also Bz = Tz = z follows from step 1, 2 we get that

\[z = Az = Bz = Sz = Tz.\]

Hence z is a common fixed point of A, B, S and T.

Uniqueness

Let z_1 and z_2 be two common fixed points of the A, B, S and T.

Then \[z_1 = Az_1 = Bz_1 = Sz_1 = Tz_1\] and \[z_2 = Az_2 = Bz_2 = Sz_2 = Tz_2.\]

Suppose \[z_1 \neq z_2.\] From (d), we have

aM(Az_1, Bz_2, t) - bM(Sz_1, Tz_2, t) \geq \big\{M(Sz_1, Tz_2, t), M(Sz_1, Az_1, t), M(Sz_1, Bz_2, t), M(Tz_2, Az_1, t), M(Tz_2, Bz_2, t)\big\}

\Rightarrow aM(z_1, z_2, t) - bM(z_1, z_2, t) \geq \big\{M(z_1, z_2, t), M(z_1, z_1, t), M(z_1, z_2, t), M(z_2, z_1, t), M(z_2, z_2, t)\big\}

\geq \big\{M(z_1, z_2, t), 1, M(z_1, z_2, t), M(z_2, z_1, t), 1\big\}

\Rightarrow (a-b) M(z_1, z_2, t) > M(z_1, z_2, t)

which is a contradiction. Hence \[z_1 = z_2.\]

Thus z is a unique common fixed point of A, B, S and T.
By Theorem (1) the self mappings $A, B, QS$ and $PT$ have a unique common fixed point i.e. $Az = Bz = QSz = PTz = z$.  

Similarly from (4) and (5)

$$\{Ax_{2n}\} \rightarrow z, \quad \{QSx_{2n}\} \rightarrow z,$$  \hspace{1cm} (8)

$$\{PTx_{2n+1}\} \rightarrow z, \quad \{Bx_{2n+1}\} \rightarrow z.$$  \hspace{1cm} (9)

By putting $x = Qz$ and $y = x_{2n+1}$ in (d) and on assuming $Qz \neq z$, we have

$$aM(AQz, Bx_{2n+1}, t) - bM(QSx_{2n}, PTx_{2n+1}, t)$$

$$\geq \phi \{M(QSx_{2n}, PTx_{2n+1}, t), M(QSx_{2n}, AQz, t), M(QSx_{2n}, Bx_{2n+1}, t), M(PTQz, AQz, t), M(PTx_{2n+1}, Bx_{2n+1}, t)\}.$$  

As $AQ = QA, QS = SQ$, we have

$$aM(QAz, Bx_{2n+1}, t) - bM(QQSz, PTx_{2n+1}, t)$$

$$\geq \phi \{M(QQSz, PTx_{2n+1}, t), M(QQSz, QAz, t), M(QQSz, Bx_{2n+1}, t), M(QPTz, QAz, t), M(PTx_{2n+1}, Bx_{2n+1}, t)\}.$$  

Taking limit as $n \rightarrow \infty$ using (6), and (9) we have

$$aM(Qz, z, t) - bM(Qz, z, t)$$

$$\geq \phi \{M(Qz, z, t), M(Qz, Qz, t), M(Qz, Qz, t), M(Qz, Qz, t), M(z, z, t)\}$$

$$\geq (a-b) M(Qz, z, t) - bM(Qz, z, t)$$

$$\Rightarrow (a-b) M(Qz, z, t) \geq \phi \{M(Qz, z, t), 2M(Qz, z, t), M(Qz, z, t), M(Qz, z, t), M(Qz, z, t)\}$$

$$\Rightarrow (a-b) M(Qz, z, t) > M(Qz, z, t)$$

which is a contradiction. Hence $Qz = z$.

Now $QSz = SQz = Sz$, gives $Sz = z = Qz$.

Again assuming $Pz \neq z$ and by putting $x = x_{2n}$ and $y = Pz$ in (4)

$$aM(Ax_{2n}, BPz, t) - bM(QSx_{2n}, PTPz)$$

$$\geq \phi \{M(QSx_{2n}, PTPz, t), M(QSx_{2n}, Ax_{2n}, t), M(QSx_{2n}, BPz, t), M(PTx_{2n}, Ax_{2n}, t), M(PTPz, BPz, t)\}.$$  

As $PT = TP$ and $BP = PB$, we get that

$$aM(Ax_{2n}, PBz, t) - bM(QSx_{2n}, PPTz, t)$$
\[ \geq \phi \{ M(QSx_{2n}, PPTz, t), M(QSx_{2n}, Ax_{2n}, t), M(QSx_{2n}, PBz, t), M(PTx_{2n}, Ax_{2n}, t), M(PPTz, PBz, t) \} \]

Taking \( n \to \infty \), we get

\[ aM(z, Pz, t) - bM(z, Pz, t) \]

\[ \geq \phi \{ M(z, Pz, t), M(z, z, t), M(z, Pz, t), M(z, z, t), M(Pz, z, t) \} \]

\[ \Rightarrow (a-b) M(z, pz, t) \geq \phi \{ M(z, pz, t), 2M(z, Pz, t), M(z, Pz, t), M(z, Pz, t), M(z, Pz, t) \} \]

\[ (a-b) M(z, Pz, t) > M(Pz, z, t) \]

which is a contradiction. Hence \( Pz = z \).

Now, \( PTz = TPz = Tz \) gives \( pz = Tz = z \).

Combining all these result, we obtain that

\[ Az = Bz = Sz = Tz = Pz = Qz = z. \]

Hence \( z \) is a common fixed point of the mapping \( A, B, S, T, P \) and \( Q \).

We can prove uniqueness of \( z \) on the same line as in theorem 5.2.

This complete the proof.

References


