Some Graph Operations on Signed Product Cordial Labeling Graphs

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ABSTRACT

A vertex labeling \( f: V(G) \rightarrow \{-1,1\} \) of a graph \( G \) with induced edge labeling \( f^*: E(G) \rightarrow \{-1,1\} \) defined by \( f^*(uv) = f(u)f(v) \) is called a signed product cordial labeling if the number of vertices with labels -1 and +1 differ at most 1 as well as the number of edges with labels -1 and +1 differs at most 1. In this paper, we prove some results on signed product cordial labeling of graphs in the context of path union at vertex of flower \( F_n \), binary tree and star. Also, we prove total graph of path \( P_n \) and \( K_{1,n} \odot G' \) (\( G' \) be the null graph with two vertices) are signed and total signed product cordial labeling graphs.

1 Introduction

We begin with simple, finite, connected and undirected graph \( G = (V(G), E(G)) \) with \( p \) vertices and \( q \) edges. For standard terminology and notations related to graph theory we refer to Gross and Yellen[8]. We will provide brief summary of definitions and other information which are prerequisites for the present study.

**Definition 1.1.** A mapping \( f: V(G) \rightarrow \{0,1\} \) is called binary vertex labeling of \( G \) and \( f(\nu) \) is called label of vertex \( \nu \) of \( G \) under \( f \). Let \( \nu_f(0) \) and \( \nu_f(1) \) be the number of vertices of \( G \) labeled with 0 and 1 respectively under \( f \). A mapping \( f^*: E(G) \rightarrow \{0,1\} \) is called binary edge labeling of \( G \). Let, \( e_f(0) \), \( e_f(1) \) the number of edges of \( G \) having labels 0 and 1 respectively under \( f^* \).

**Definition 1.2.** A binary vertex labeling of graph \( G \) with induced edge labeling \( f^*: E(G) \rightarrow \{0,1\} \) defined by \( f^*(e = uv) = |f(u)f(v)| \) is called cordial labeling if \( |\nu_f(0) - \nu_f(1)| \leq 1 \) and \( |e_f^*(0) - e_f^*(1)| \leq 1 \). A graph \( G \) is cordial if it admits a cordial labeling.

The concept of cordial labeling was introduced by Cahit[5] as a weaker version of graceful and harmonious labeling. This concept is explored by many researchers like Andar et.al.[1,2], Vaidya and Dani [14,15]. Vaidya and Vihol [14] have proved that middle graphs of path, crown, \( K_{1,n} \), Tadpole \( T(n,1+l) \) are cordial graph. Vaidya and Shah[17,18] have discussed cordial labeling of some bistar related graphs and cordial labeling of snake related graphs. Motivated through the concept of cordial labeling Babujee and Shobana [3] introduced the concepts of cordial languages and cordial numbers. Lawrence and Koilraj [10] have discussed cordial labeling for the splitting graph of some standard graphs.
Definition 1.3. (Babujee and Shobana [4]) A vertex labeling \( f : V(G) \rightarrow \{-1, 1\} \) of a graph \( G \) with induced edge labeling \( f^* : E(G) \rightarrow \{-1, 1\} \), defined by \( f^*(uv) = f(u)f(v) \), is called signed product cordial labeling if \( |v_j(-1) - v_j(1)| \leq 1 \) and \( |e_{f^*}(-1) - e_{f^*}(1)| \leq 1 \). A graph \( G \) is signed product cordial if it admits a signed product cordial labeling.

Santhi and James Albert [12] have introduced the concept of total signed product cordial labeling.

Definition 1.4. Let \( f : V(G) \rightarrow \{-1, 1\} \) with induced edge labeling \( f^* : E(G) \rightarrow \{-1, 1\} \) defined by \( f^*(uv) = f(u)f(v) \). Then \( f \) is called a total signed product cordial labeling if \( |(v_j(-1) + e_{f^*}(-1)) - (v_j(1) + e_{f^*}(1))| \leq 1 \). A graph with a total signed product cordial labeling is called a total signed product cordial graph.

Definition 1.5. The total graph \( T(G) \) of a graph \( G \) is the graph whose vertex set is \( V \cup E \), with two vertices of \( T(G) \) being adjacent if and only if the corresponding elements of \( G \) are adjacent or incident.

Definition 1.6. (Shee and Ho [13]) Let \( G_1, G_2, \ldots, G_n \) be \( n \) copies of a fixed graph \( G \). The graph \( G \) obtained by adding an edge between \( G_i \) and \( G_{i+1} \) for \( i = 1, 2, \ldots, n-1 \) is called path union of \( G \) and is defined by \( G(n) \).

Definition 1.7. Let \( G_1, G_2, \ldots, G_n \) be \( n \) copies of a graph \( G \). The graph \( G_u(n) \) obtained by adding edge between the vertex \( u \) in \( G_i \) and the copy of \( u \) in \( G_{i+1} \) for \( i = 1, 2, \ldots, n-1 \), we call \( G_u(n) \) is called the path union of \( G \) at \( u \).

Definition 1.8. (Frucht and Harary [7]) Let \( G_1 \) and \( G_2 \) be two simple, connected graphs. The corona \( G_1 \Theta G_2 \) of two graphs \( G_1 \) and \( G_2 \) is the graph obtained by taking one copy of \( G_1 \) (which has \( p_1 \) points) and \( p_1 \) copies of \( G_2 \), and then joining the \( i^{th} \) copy of \( G_2 \) with \( G_1 \).

2 MAIN RESULTS

Theorem 2.1. Let \( G = F_n \) and \( w \) be the centre vertex of \( F_n \). Then \( G_w(2) \) is signed product cordial labeling graph.

Proof

Let \( u \) and \( v \) be the centre vertices of the flower graphs \( G_i \) and \( G_2 \).

Let us denote the vertices in the cycle of \( G_1 \) and \( G_2 \) as \( u_i \) and \( v_i \) (1 ≤ \( i \) ≤ \( n \)) respectively.

Let the end vertices of \( G_2 \) and \( G_2 \) as \( u'_i \) and \( v'_i \) (1 ≤ \( i \) ≤ \( n \)) respectively.

Let \( w_1, w_2, \ldots, w_k \) be the vertices of path \( P_k \) with \( u = w_1 \) and \( v = w_k \) \( k \geq 2 \).

Then \( |V(G_w(2))| = 4n + k \) and \( |E(G_w(2))| = 8n + k - 1 \).

Define \( f : V(G_w(2)) \rightarrow \{-1, 1\} \) as given below.
\[
\begin{align*}
f(u) &= -1 \\
f(v) &= 1 \\
f(u_i) &= f(v_i) = 1 \\
f(v_i) &= f(u_i) = -1 \\
f(w_i) &= \begin{cases} -1, & (1, 2) \text{ (mod 4)}, \\ 1, & (0, 3) \text{ (mod 4)}. \end{cases}
\end{align*}
\]

Case(i) Suppose \( k = 2 \).

\[
\begin{align*}
v_f(-1) &= v_f(1) = 2n + 1 \\
e_{f^*}(-1) &= e_{f^*}(1) = 4n + 1
\end{align*}
\]

Case(ii) Suppose \( k \equiv 2 \text{ (mod 4)} \) and \( k \neq 2 \).

\[
\begin{align*}
v_f(-1) &= v_f(1) = 2n + 3 \\
e_{f^*}(-1) &= 4n + 3 \\
e_{f^*}(1) &= 4n + 2
\end{align*}
\]

Case(iii) Suppose \( k \equiv 1 \text{ (mod 4)} \).

\[
\begin{align*}
v_f(-1) &= 2n + 3 \\
v_f(1) &= 2n + 2 \\
e_{f^*}(-1) &= e_{f^*}(1) = 4n + 2
\end{align*}
\]

Case(iv) Suppose \( k \equiv 3 \text{ (mod 4)} \).

\[
\begin{align*}
v_f(-1) &= 2n + 2 \\
v_f(1) &= 2n + 1 \\
e_{f^*}(-1) &= e_{f^*}(1) = 4n + 1
\end{align*}
\]

Case(v) Suppose \( k \equiv 0 \text{ (mod 4)} \).

\[
\begin{align*}
v_f(-1) &= v_f(1) = 2n + 2 \\
e_{f^*}(-1) &= 4n + 1 \\
e_{f^*}(1) &= 4n + 2
\end{align*}
\]

By all the above cases, we have

\[
|v_f(-1) - v_f(1)| \leq 1 \text{ and } |e_{f^*}(-1) - e_{f^*}(1)| \leq 1.
\]
Hence, G is a signed product cordial graph.

Example 2.2. Suppose \( G = F_3 \) and \( w \) be the centre vertex of G. Then \( G_w(2) \) with its signed product cordial labeling.

\[
\text{Fig. 1.}
\]

Theorem 2.3. Path union of two copies of binary tree at the root vertex is signed product cordial graph.

Proof

Let G be the path union of two copies binary tree at the root vertex with \( m (m \geq 1) \) levels.

Let \( G_1 \) and \( G_2 \) be two copies of binary tree with \( m \) levels.

Let \( u \) and \( v \) be the root of \( G_1 \) and \( G_2 \) respectively.

Clearly \( i^{th} \) level of \( G_1 \) and \( G_2 \) has \( 2^i \) vertices.

Then \( |V(G_i)| = 2^{m+1} - 1 \) and \( |E(G_i)| = 2^{m+1} - 2 \) (\( i=1,2 \))

Let \( u_i \) and \( v_i \) be the vertices in the \( i^{th} \) level of \( G_1 \) and \( G_2 \) respectively. (\( i = 1, 2, \ldots, 2^{i+1} - 1 \)).

Let \( w_i \) be the vertices of path \( P_k (k \geq 2) \) then \( w_1 = u \) and \( w_n = v \).

Define \( f: V(G) \rightarrow \{-1, 1\} \) as given below.
By the above labeling pattern, we have

\[ f(u_i) = \begin{cases} 
  1, & \text{i is odd,} \\
  -1, & \text{i is even.}
\end{cases} \]

\[ f(v_i) = \begin{cases} 
  1, & \text{i is odd,} \\
  -1, & \text{i is even.}
\end{cases} \]

\[ f(w_i) = \begin{cases} 
  -1, & \text{i \equiv (1, 2) (mod 4),} \\
  1, & \text{i \equiv (0, 3)(mod 4).}
\end{cases} \]

\[ f(w_n) = 1 \quad \text{if} \quad k \equiv 2 \quad \text{(mod 4)} \]

Case(i) Suppose \( k = 2 \).

\[ v_f(-1) = v_f(1) = \frac{|V(G)|}{2} \]

\[ e_{r'}(-1) = \frac{|V(G)|}{2} \]

\[ e_{r'}(1) = \frac{|V(G)|}{2} - 1 \]

Case(ii) Suppose \( k \equiv 0 \quad \text{(mod 4)} \).

\[ v_f(-1) = v_f(1) = \frac{|V(G)|}{2} \]

\[ e_{r'}(1) = \frac{|V(G)|}{2} \]

\[ e_{r'}(-1) = \frac{|V(G)|}{2} - 1 \]

Case(ii) Suppose \( k \equiv (1, 3) \quad \text{(mod 4)} \).

\[ v_f(-1) = \frac{|V(G)| + 1}{2} \]

\[ v_f(1) = \frac{|V(G)| - 1}{2} \]

\[ e_{r'}(-1) = e_{r'}(1) = \frac{|V(G)|}{2} - 1 \]

Case(ii) Suppose \( k \equiv 2 \quad \text{(mod 4)} \) and \( k \neq 2 \).

\[ v_f(-1) = v_f(1) = \frac{|V(G)|}{2} \]

\[ e_{r'}(-1) = \frac{|V(G)|}{2} \]

\[ e_{r'}(1) = \frac{|V(G)|}{2} - 1 \]
By all the above cases, we have

\[ |v_f(-1) - v_f(1)| \leq 1 \text{ and } |e_f(-1) - e_f(1)| \leq 1. \]

Hence, \( G \) is signed product cordial labeling graph.

**Example 2.4.**

The graph obtained by joining two copies of binary tree (7 vertices) with its signed product cordial labeling.

\[ \text{Fig. 2.} \]

**Theorem 2.5.** Path union of two copies of binary tree at the root vertex is a total signed product cordial graph.

Proof

Let \( G \) be the graph obtained by joining two copies of binary tree by path \( P_k \), labeling the vertices as in the above theorem 2.3., we have

\[ |(v_f(-1) + e_f(-1)) - (v_f(1) + e_f(1))| \leq 1. \]

Hence, \( G \) is a total signed product cordial graph.

**Theorem 2.6.** Path union of two copies of \( K_{1,n} \) by \( P_2 \) is a signed product cordial graph.

Proof

Let \( G \) be the path union of two copies of \( K_{1,n} \) by \( P_2 \).

Let \( u \) and \( v \) be the centre vertices of \( G_1 \) and \( G_2 \) respectively.

Let \( u_i \) and \( v_i \) be the remaining vertices of \( G_1 \) and \( G_2 \) respectively. \((i = 1,2, \ldots, n)\)

Then \(|V(G_i)| = n + 1\) and \(|E(G_i)| = n \ (i = 1, 2)\).
Let \( w_1 \) be the vertices of path \( P_2 \) with \( w_1 = u \) and \( w_2 = v \).

Define \( f : V(G) \rightarrow \{-1, 1\} \) as given below.

\[
\begin{align*}
  f(u) &= -1 \\
  f(w_1) &= \begin{cases} 
    -1, & i \text{ is odd} \\
    1, & i \text{ is even}
  \end{cases} \\
  f(w_2) &= 1
\end{align*}
\]

Case (i) Suppose \( n \equiv 1, 3 \pmod{4} \).

\[
\begin{align*}
  v_f(-1) &= v_f(1) = n + 1 \\
  e_f'(1) &= n + 1
\end{align*}
\]

Case (ii) Suppose \( n \equiv 0, 2 \pmod{4} \).

\[
\begin{align*}
  v_f(-1) &= v_f(1) = n + 1 \\
  e_f'(1) &= n + 1
\end{align*}
\]

Therefore, \( |v_f(-1) - v_f(1)| \leq 1 \) and \( |e_f'(1) - e_f'(1)| \leq 1 \).

Hence, \( G \) is a signed product cordial graph.
Example 2.7. Path union of two copies of star by $P_2$ with its signed product cordial labeling

\begin{figure}
\centering
\includegraphics[width=\textwidth]{example27}
\caption{Example 2.7. Path union of two copies of star by $P_2$ with its signed product cordial labeling}
\end{figure}

Theorem 2.8. Path union of two copies of star by $P_2$ is a total signed product cordial graph.

Proof

Let $G$ be the path union of two copies of $K_{1,n}$ by $P_2$. Labeling the vertices as in the above theorem 2.6., we have

Case (i) Suppose $n \equiv (1,3) \pmod{4}$.

\[ v_f(-1) + e_f^*(-1) = 2n + 1 \]
\[ v_f(1) + e_f^*(1) = 2n + 2 \]

Case (ii) Suppose $n \equiv (0,2) \pmod{4}$.

\[ v_f(-1) + e_f^*(-1) = 2n + 2 \]
\[ v_f(1) + e_f^*(1) = 2n + 1 \]

By the above cases, we have

\[ |(v_f(-1) + e_f^*(-1)) - (v_f(1) + e_f^*(1))| = 1 \]

Therefore, \[ |(v_f(-1) + e_f^*(-1)) - (v_f(1) + e_f^*(1))| \leq 1 \].

Hence, $G$ is a total signed product cordial graph.

Theorem 2.9. $T(P_n)$ is a total signed product cordial graph.
Proof

Let $G = T(P_n)$ be the total graph of $P_n$.

Let $v_i$ and $e_j$ be vertices and edges of $P_n$ ($i = 1, 2, \ldots, n$) and ($j = 1, 2, \ldots, n-1$)

Then $|V(G)| = 2n - 1$ and $|E(G)| = 2n + 1$.

Define $f : V(G) \rightarrow \{-1, 1\}$ as given below.

$$
\begin{align*}
 f(v_i) &= \begin{cases} 
 -1, & 1 \leq i \leq n, \\
 1, & \text{otherwise.}
\end{cases} \\
 f(e_j) &= \begin{cases} 
 1, & 1 \leq i \leq n - 1, \\
 -1, & \text{otherwise.}
\end{cases}
\end{align*}
$$

By the above labeling pattern, we have

$$
\begin{align*}
 v_f(-1) &= n + 1 \\
 v_f(1) &= n \\
 e_f^*(-1) &= 2n \\
 e_f^*(1) &= n + 1.
\end{align*}
$$

Thus, we have

$$
\left| (v_f(-1) + e_f^*(-1)) - (v_f(1) + e_f^*(1)) \right| \leq 1.
$$

Hence, $T(P_n)$ is a total signed product cordial graph.

**Example 2.10**

$T(P_n)$ with its total signed product cordial labeling.

![Diagram](http://www.ijmttjournal.org)
Theorem 2.11. \( G = \left( K_{1,n} \odot G \right) \) is a signed product cordial graph.

Proof

Let \( G = \left( K_{1,n} \odot G \right) \) be a graph with \( V(G) = \{u, u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n, w_1, w_2, \cdots, w_n\} \) and \( E(G) = \{uu_i, 1 \leq i \leq n\} \cup \{v_iv_i, 1 \leq i \leq n\} \cup \{u_iw_i, 1 \leq i \leq n\} \).

Define \( f : V(G) \rightarrow \{-1, 1\} \) as given below.

\[
f(u) = -1.
\]

\[
f(u_i) = \begin{cases} 
1, & \text{if } i \text{ is odd}, \\
-1, & \text{if } i \text{ is even}.
\end{cases}
\]

\[
f(v_i) = 1, \quad 1 \leq i \leq n
\]

\[
f(w_i) = -1, \quad 1 \leq i \leq n
\]

Case (i) Suppose \( n \) is odd.

\[
v_f(-1) = v_f(1) = \frac{3n + 3}{2}
\]

\[
e_f(-1) = \frac{3n + 3}{2}
\]

\[
e_f(1) = \frac{3n + 1}{2}
\]

Case (ii) Suppose \( n \) is even.

\[
v_f(-1) = \frac{3n + 2}{2}
\]

\[
v_f(1) = \frac{3n + 4}{2}
\]

\[
e_f(-1) = e_f(1) = \frac{3n + 2}{2}
\]

By the above cases, we have

\[
|v_f(-1) - v_f(1)| \leq 1 \quad \text{and} \quad |e_f(-1) - e_f(1)| \leq 1.
\]

Hence, \( G = \left( K_{1,n} \odot G \right) \) is a signed product cordial graph.

Example

\( G = \left( K_{1,7} \odot G \right) \) with its signed product cordial graph.
**Theorem 2.12.** \( G = (K_{1,n} \odot G') \) is a total signed product cordial graph.

**Proof**

Let \( G = (K_{1,n} \odot G') \) be a graph with labeling pattern as in the above theorem 2.11., we have

Case (i) Suppose \( n \) is odd.

\[
\nu_f(-1) + e_f^*(-1) = 3n + 3
\]

\[
\nu_f(1) + e_f^*(1) = 3n + 2
\]

Case (ii) Suppose \( n \) is even.

\[
\nu_f(-1) + e_f^*(-1) = 3n + 2
\]

\[
\nu_f(1) + e_f^*(1) = 3n + 3
\]

Therefore, \( |(\nu_f(-1) + e_f^*(-1)) - (\nu_f(-1) + e_f^*(-1))| \leq 1 \).

Hence, \( G = (K_{1,n} \odot G') \) is a total signed product cordial graph.
References

[3]. J.Baskar Babujee and Shobana Loganathan, Cordial languages and Cordial Numbers, Journals of Applied Computer Science and Mathematics.