Anti Q-fuzzy PMS- ideals in PMS-algebras

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Abstract: In this paper, we introduce the concept of Anti Q-fuzzy PMS-ideals of PMS-algebras, lower level cuts of a fuzzy set and proved some results. We discussed few results of anti Q-fuzzy PMS-ideals of PMS-algebras in homomorphism and Cartesian product.

Keywords: PMS-algebra, fuzzy PMS- ideal, Anti Q-fuzzy PMS-ideal, lower level cuts, homomorphism, Cartesian product.

1. Introduction

The concept of fuzzy set was introduced by L.A.Zadeh in 1965 [19]. Since then these ideas have been applied to other algebraic structures such as groups, rings, modules, vector spaces and topologies. K.Iseki and S.Tanaka [2] introduced the concept of BCK-algebras in 1978 and K.Iseki [3] introduced the concept of BCI-algebras in 1980. It is known that the class of BCK –algebras is a proper subclass of the class of BCI algebras in 2002. P.M.Sithar Selvam and K.T.Nagalakshmi [5,6] introduced the concept of PMS-algebras , which is a generalization of BCK / BCI / TM / KUS / PS algebras. R.Biswas[1] introduced the concept of Anti fuzzy subgroups of groups. Modifying his idea, in this paper we applied the idea in PMS-algebras. We introduced the notion of Anti Q- fuzzy PMS-ideals of PMS-algebras and investigate how to deal with the homomorphic, Anti homomorphic and inverse image of Anti Q-fuzzy PMS-ideals of PMS-algebras.

2. Preliminaries

In this section we site the fundamental definitions that will be used in the development of this paper.

Definition 2.1 :[5,6] A nonempty set X with a constant 0 and a binary operation ‘ * ’ is called PMS – algebra if it satisfies the following axioms.
1. 0 * x = x  
2. (y * x) * (z * x) = z * y , ∀ x , y, z ∈ X.

In X, we define a binary relation ≤ by : x ≤ y if and only if x * y = 0.

Definition 2.2:[5,6] Let X be a PMS - algebra and I be a subset of X, then I is called a PMS-ideal of X if it satisfies the following conditions:
1. 0 ∈ I  
2. z * y ∈ I and z * x ∈ I ⇒ y * x ∈ I for all x, y, z ∈ X.

Example 2.3:[5,6] Let X = {0, a, b, c} be the set with the following table.

<table>
<thead>
<tr>
<th>*</th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>a</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>a</td>
<td>b</td>
<td>0</td>
<td>a</td>
<td>b</td>
</tr>
<tr>
<td>b</td>
<td>a</td>
<td>b</td>
<td>0</td>
<td>c</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>c</td>
<td>a</td>
<td>0</td>
</tr>
</tbody>
</table>

Then (X, * , 0 ) is a PMS – algebra and I = {0,a,b} is a PMS-ideal.

Definition 2.4:[7,8] : Let S be a non empty subset of a PMS -algebra X , then S is called a PMS-sub algebra of X if x * y ∈ S ,for all x, y ∈ S.

Definition 2.5 [17,19] : Let X be a non-empty set. A fuzzy subset μ of the set X is a mapping μ : X →[0, 1].
Definition 2.6 [11,12,13]: Let Q and G be any two sets. A mapping \( \beta: G \times Q \rightarrow [0, 1] \) is called a Q–fuzzy set in G.

**MAIN RESULTS**

3. ANTI Q-FUZZY PMS – IDEALS OF PMS ALGEBRAS

**Definition 3.1**: A Q-fuzzy set \( \mu \) in X is called a Q-fuzzy PMS-ideal of X if

1. \( \mu(0, q) \geq \mu(x, q) \)
2. \( \mu(y \ast x, q) \geq \min \{ \mu(z \ast y, q), \mu(z \ast x, q) \} \), for all \( x, y, z \in X \) and \( q \in Q \).

**Definition 3.2**: A Q-fuzzy set \( \mu \) of X is called an anti Q-fuzzy PMS-ideal of X if

1. \( \mu(0, q) \leq \mu(x, q) \)
2. \( \mu(y \ast x, q) \leq \max \{ \mu(z \ast y, q), \mu(z \ast x, q) \} \), for all \( x, y, z \in X \) and \( q \in Q \).

**Theorem 3.1**: Every Anti Q-fuzzy PMS-ideal \( \mu \) of a PMS-algebra X is order preserving.

**Proof**: Let \( \mu \) be an anti Q-Fuzzy PMS-ideal of a PMS-algebra X and let \( x, y \in X \) and \( q \in Q \) be such that \( x \leq y \), then \( x \ast y = 0 \).

Now \( \mu(x, q) = \max \{ \mu(0 \ast x, q) \} \)

\[ \leq \max \{ \mu(z \ast 0, q), \mu((x \ast y) \ast (z \ast y), q) \} \]

\[ = \max \{ \mu(z \ast 0, q), \mu(0 \ast (z \ast y), q) \} \]

\[ = \max \{ \mu(0 \ast 0, q), \mu(0 \ast y, q) \} \]

(Taking \( z = 0 \))

\[ = \max \{ \mu(0, q), \mu(y, q) \} = \mu(y, q) \]

Thus \( \mu(x, q) \leq \mu(y, q) \).

**Theorem 3.2**: \( \mu \) is a Q-fuzzy PMS-ideal of a PMS-algebra X iff \( \mu^c \) is an anti Q-fuzzy PMS-ideal of X.

**Proof**: Let \( \mu \) be a Q-Fuzzy PMS-ideal of X and let \( x, y, z \in X \) and \( q \in Q \).

\[ \mu(0, q) \geq \mu(x, q) \]

\[ 1 - \mu^c(0, q) \geq 1 - \mu^c(x, q) \]

\[ \mu^c(0, q) \leq \mu^c(x, q) \]

and \( \mu^c(y \ast x, q) = 1 - \mu(y \ast x, q) \)

\[ \leq 1 - \min \{ \mu(z \ast y, q), \mu(z \ast x, q) \} \]

\[ = 1 - \min \{ 1 - \mu^c(z \ast y, q), 1 - \mu^c(z \ast x, q) \} \]

\[ = \max \{ \mu^c(z \ast y, q), \mu^c(z \ast x, q) \} \]

Thus \( \mu^c \) is an anti Q-fuzzy PMS-ideal of X. The converse also can be proved similarly.

**Theorem 3.3**: Let X be a PMS-algebra. For any anti Q-fuzzy PMS-ideal \( \mu \) of X, \( N_\mu = \{ x \in X \) and \( q \in Q / \mu(x, q) = \mu(0, q) \) \} is a PMS-ideal of X.

**Proof**: Let \( z \ast y, z \ast x \in N_\mu \). Then \( \mu(z \ast y, q) = \mu(z \ast x, q) = \mu(0, q) \)

Since \( \mu \) is an anti Q-fuzzy PMS-ideal of X,

\[ \mu(y \ast x, q) \leq \max \{ \mu(z \ast x, q), \mu(z \ast y, q) \} \]

\[ = \max \{ \mu(0, q), \mu(0, q) \} \]

\[ = \mu(0, q) \]

Hence \( y \ast x \in N_\mu \). Therefore \( N_\mu \) is a PMS-ideal of X.

**Theorem 3.4**: If \( \lambda \) and \( \mu \) are anti Q-fuzzy PMS ideals of a PMS-algebra X, then \( \lambda \cap \mu \) is also an anti Q-fuzzy PMS-ideal of X.

**Proof**: Let \( x, y, z \in X \) and \( q \in Q \). Then

\[ (\lambda \cap \mu)(0, q) = \min \{ \lambda(0, q), \mu(0, q) \} \]

\[ \leq \min \{ \lambda(x, q), \mu(x, q) \} \]
\[(\lambda \cap \mu)(y^*,q) = \min \{\lambda(y^*,q), \mu(y^*,q)\}\]
\[\leq \min \{\max \{\lambda(z^*,x), \lambda(x,y), q\}, \max \{\mu(z^*,x), \mu(z,y), q\}\}\]
\[= \min \{\max \{\lambda(z^*,x), \mu(z^*,x)\}, \max \{\lambda(z,y), q\}, \max \{\mu(z,y), q\}\}\]
\[\leq \max \{\min \{\lambda(z^*,x), \mu(z^*,x)\}, \min \{\lambda(z,y), q\}, \max \{\mu(z,y), q\}\}\]
\[\Rightarrow (\lambda \cap \mu)(y^*,t) \leq \max \{\lambda(\cap \mu)(z^*,y), q\}, \lambda(\cap \mu)(z^*,x), q\}.\]
\[\Rightarrow (\lambda \cap \mu) is also an anti Q-fuzzy PMS ideal of X.\]

**Theorem 3.5:** The union of any set of anti Q-fuzzy PMS-ideals in PMS-algebra X is also an anti Q-fuzzy PMS-ideal.

**Proof:** Let \(\mu_i\) be a family of anti Q-fuzzy PMS-ideals of PMS-algebras X.

Then for any \(x, y, z \in X\) and \(q \in Q\),
\[(\cup \mu_i)(0,q) = \sup \{\mu_i(0,q)\}\]
\[\leq \sup \{\mu_i(x,q)\} = (\cup \mu_i)(x,q)\]
\[(\cup \mu_i)(y^*,q) = \sup \{\mu_i(y^*,q)\}\]
\[\leq \sup \{\max \{\mu_i(z^*,y,q), \mu_i(z^*,x,q)\}\} = \max \{\sup \{\mu_i(z^*,y,q)\}, \sup \{\mu_i(z^*,x,q)\}\}\]
\[\leq \max \{(\cup \mu_i)(z^*,y,q), (\cup \mu_i)(z^*,x,q)\}\]

This completes the proof.

**Definition 3.6:** Let \(\mu\) be a Q-fuzzy set of X. For a fixed \(t \in [0,1]\), the set \(\mu_t = \{x \in X \mid \mu(x,q) \leq t \text{ for all } q \in Q\}\) is called the lower level subset of \(\mu\). Clearly \(\mu^t = \cup \mu_i = X\) for \(t \in [0,1]\) if \(t_1 < t_2\), then \(\mu_{t_1} \subseteq \mu_{t_2}\).

**Theorem 3.7:** If \(\mu\) is an anti Q-fuzzy PMS-ideal of PMS-algebra X, then \(\mu_t\) is a PMS-ideal of X for every \(t \in [0,1]\).

**Proof:** Let \(\mu\) be an anti Q-fuzzy PMS-ideal of PMS-algebra X.

Clearly \(0 \in \mu_t\).

Let \(z^* x \in \mu_t\) and \(z^* y \in \mu_t\), for all \(x, y \in X\) and \(q \in Q\).
\[\Rightarrow \mu(z^* x,q) \leq t\] and \(\mu(z^* y, q) \leq t\).
\[\mu(y^*, q) \leq \max \{\mu(z^* y, q), \mu(z^* x, q)\} \leq \max \{t, t\} = t.\]
\[\Rightarrow y^* x \in \mu_t.\]

Hence \(\mu_t\) is an PMS-ideal of X for every \(t \in [0,1]\).

**Theorem 3.8:** Let \(\mu\) be a Q-fuzzy set of PMS-algebra X. If for each \(t \in [0,1]\), the lower level cut \(\mu_t\) is a PMS-ideal of X, then \(\mu\) is an anti Q- fuzzy PMS-ideal of X.

**Proof:** Let \(\mu\) be a PMS-ideal of X.

If \(\mu(0,q) > \mu(x,q)\) for some \(x \in X\) and \(q \in Q\), then \(\mu(0,q) > t_0 > \mu(x,q)\) by taking
\[t_0 = \frac{1}{2} \{\mu(0,q) + \mu(x,q)\}.\]

Hence \(0 \notin \mu_{t_0}\) and \(x \in \mu_{t_0}\), which is a contradiction.

Therefore, \(\mu(0,q) \leq \mu(x,q)\).

Let \(x, y, z \in X\) and \(q \in Q\) be such that \(\mu(y^* x, q) \geq \max \{\mu(z^* y, q), \mu(z^* x, q)\}\).

Taking \(t_1 = \frac{1}{2} \{\mu(y^* x, q) + \mu(z^* y, q), \mu(z^* x, q)\}\)
\[\Rightarrow \mu(y^* x, q) > t_1 \geq \max \{\mu(z^* y, q), \mu(z^* x, q)\}.\]

It follows that \(z^* y, z^* x \in \mu_t\) and \(y^* x \notin \mu_t\). This is a contradiction.

Hence \(\mu(y^* x, q) \leq \max \{\mu(z^* y, q), \mu(z^* x, q)\}\)
Therefore $\mu$ is an anti Q-fuzzy PMS-ideal of $X$.

4. HOMOMORPHISM AND ANTI HOMOMORPHISM ON ANTI Q-FUZZY PMS- ALGEBRAS

In this section, we discussed about ideals in PMS-algebra under homomorphism and anti homomorphism and some of its properties.

**Definition 4.1 :** Let $(X, *, 0)$ and $(Y, \Delta, 0)$ be PMS-algebras. A mapping $f : X \to Y$ is said to be a homomorphism if $f(x * y) = f(x) * f(y)$ for all $x, y \in X$.

**Definition 4.2 :** Let $(X, *, 0)$ and $(Y, \Delta, 0)$ be PMS-algebras. A mapping $f : X \to Y$ is said to be an anti homomorphism if $f(x * y) = f(y) \Delta f(x)$ for all $x, y \in X$.

**Definition 4.3 :** Let $f : X \to X$ be an endomorphism and $\mu$ be a fuzzy set in $X$. We define a new fuzzy set in $X$ by $\mu_f$ in $X$ as $\mu_f(x) = \mu(f(x))$ for all $x \in X$.

**Theorem 4.4 :** Let $f$ be an endomorphism of a PMS-algebra $X$. If $\mu$ is an anti Q-fuzzy PMS-ideal of $X$, then so is $\mu_f$.

**Proof:** Let $\mu$ be an anti Q-fuzzy PMS-ideal of $X$.

Now, $\mu_f(0, q) = \mu(f(0, q)) = \mu(f(x, q)) = \mu(x, q)$, for all $x, y \in X$ and $q \in Q$.

Let $x, y, z \in X$ and $q \in Q$.

Then $\mu_f(y * x, q) = \mu(f(y * x, q)) = \mu(f(y, q) * f(x, q))$.

Hence $\mu_f$ is an anti Q-fuzzy PMS-ideal of $X$.

**Theorem 4.5 :** Let $f : X \to Y$ be an epimorphism of PMS-algebra. If $\mu_f$ is an anti Q-fuzzy PMS-ideal of $X$, then $\mu$ is an anti Q-fuzzy PMS-ideal of $Y$.

**Proof:** Let $\mu_f$ be an anti Q-fuzzy PMS-ideal of $X$.

Let $y \in Y$ and $q \in Q$. Then there exists $x \in X$ such that $f(x, q) = (y, q)$.

Now, $\mu(0, q) = \mu(f(0, q)) = \mu_f(0, q) = \mu_f(x, q)$.

Hence $\mu$ is an anti Q-fuzzy PMS-ideal of $Y$.

**Theorem 4.6 :** Let $f : X \to Y$ be a homomorphism of PMS-algebra. If $\mu$ is an anti Q-fuzzy PMS-ideal of $Y$ then $\mu_f$ is an anti Q-fuzzy PMS-ideal of $X$. 


5. CARTESIAN PRODUCT OF ANTI Q-FUZZY PMS-IDEALS OF PMS–ALGEBRAS

In this section, we introduce the concept of Cartesian product of anti Q-fuzzy PMS-ideals of PMS-algebra.

Definition 5.1: Let \( \mu \) and \( \Delta \) be the fuzzy sets in \( X \). The Cartesian product \( \mu \times \Delta : X \times X \rightarrow [0,1] \) is defined by

\[
(\mu \times \Delta)(x,y) = \min \{\mu(x), \Delta(y)\}
\]

for all \( x, y \in X \).

Definition 5.2: Let \( \mu \) and \( \Delta \) be the fuzzy sets in \( X \). The Cartesian product \( \mu \times \Delta : X \times X \rightarrow [0,1] \) is defined by

\[
(\mu \times \Delta)(x,y) = \max \{\mu(x), \Delta(y)\}
\]

for all \( x, y \in X \).

Definition 5.3: Let \( \mu \) and \( \Delta \) be the fuzzy sets in \( X \). The Cartesian product \( \mu \times \Delta : X \times X \rightarrow [0,1] \) is defined by

\[
(\mu \times \Delta)(x,y) = \max \{\mu(x), \Delta(y)\}
\]

for all \( x, y \in X \) and \( q \in Q \).

Theorem 5.4: If \( \mu \) and \( \Delta \) are anti Q-fuzzy PMS-ideals in a PMS– algebra \( X \), then \( \mu \times \Delta \) is an anti Q-fuzzy PMS-ideal in \( X \times X \).

Proof: Let \( (x_1, x_2) \in X \times X \) and \( q \in Q \).

\[
(\mu \times \Delta)((x_1, 0, x_2, 0), q) = \max \{\mu(x_1, 0, q), \Delta(x_2, 0, q)\}
\]

\[
= \max \{\mu(x_1, q), \Delta(x_2, q)\}
\]

\[
= (\mu \times \Delta)((x_1, x_2), q)
\]

\[
\therefore (\mu \times \Delta)((x_1, 0, x_2, 0), q) \leq (\mu \times \Delta)((x_1, x_2), q)
\]

Let \( (x_1, x_2), (y_1, y_2), (z_1, z_2) \in X \times X \).

\[
(\mu \times \Delta)((y_1, y_2),(x_1, x_2), q) = (\mu \times \Delta)((y_1, y_2), (x_1, x_2), q)
\]

\[
= \max \{\mu(y_1, x_1, q) \Delta(y_2, x_2, q)\}
\]

\[
= \max \{\mu(z_1, x_1, q) \Delta(z_2, x_2, q)\}
\]

\[
= \max \{\mu(z_1, x_1, q) \Delta(z_2, x_2, q)\}
\]

Hence, \( \mu \times \Delta \) is an anti Q-fuzzy PMS-ideal in \( X \times X \).

Theorem 5.5: Let \( \mu \) and \( \Delta \) be fuzzy sets in a PMS-algebra \( X \) such that \( \mu \times \Delta \) is an anti Q-fuzzy PMS-ideal of \( X \times X \). Then

(i) Either \( \mu(0, q) \leq \mu(x, q) \) (or \( \Delta(0, q) \leq \Delta(x, q) \) for all \( x \in X \) and \( q \in Q \).

(ii) If \( \mu(0, q) \leq \mu(x, q) \) for all \( x \in X \) and \( q \in Q \), then either \( \Delta(0, q) \leq \Delta(x, q) \) (or \( \Delta(0, q) \leq \Delta(x, q) \) for all \( x \in X \) and \( q \in Q \).

(iii) If \( \Delta(0, q) \leq \Delta(x, q) \) for all \( x \in X \) and \( q \in Q \), then either \( \mu(0, q) \leq \mu(x, q) \) (or \( \mu(0, q) \leq \mu(x, q) \).
Proof: Straightforward.

Theorem 5.6: Let μ and δ be fuzzy sets in a PMS-algebra X such that μ x δ is an anti Q-fuzzy PMS-ideal of X x X. Then either μ or δ is an anti Q-fuzzy PMS-ideal of X.

Proof: First we prove that δ is an anti Q- fuzzy PMS-ideal of X.

Since by 5.5(i) either μ(0,q) ≤ μ(x, q) or δ(0,q) ≤ δ(x, q) for all x ∈ X and q ∈ Q.

Assume that δ(0,q) ≤ δ(x, q) for all x ∈ X and q ∈ Q. It follows from 6.2(iii) that either μ(0,q) ≤ μ(x,q) (or) μ(0,q) ≤ δ(x,q).

If μ(0,q) ≤ δ(x,q), for any x ∈ X and q ∈ Q . then

\[ \delta(x, q) = \max \{ \mu(0,q), \delta(x,q) \} = (\mu \times \delta)(0, x, q) \]

\[ \delta(y \ast x, q) = (\mu \times \delta)(0, y \ast x, q) \leq \max \{(\mu \times \delta)(0, z \ast y, q), (\mu \times \delta)(0, z, q) \} \]

\[ = \max \{(\mu \times \delta)(0, z \ast y, q), (\mu \times \delta)(0, z, q) \} \]

\[ = \max \{ \delta(z \ast y, q), \delta(z, q) \} \]

\[ \Rightarrow \delta(y \ast x, q) \leq \max \{ \delta(z \ast y, q), \delta(z, q) \} \]

Hence δ is an anti fuzzy PMS-ideal of X.

Similarly we will prove that μ is an anti Q- fuzzy PMS-ideal of X.

6. CONCLUSION

In this article we have discussed anti Q-fuzzy PMS-ideal of PMS-algebras and its lower level cuts in detail. We hope that this work would lay other foundations for further study of the theory of PMS-algebras. In our future study of fuzzy structure of PMS-algebra, can be extended to the topics, intuitionistic fuzzy sets, interval valued fuzzy sets, for more interesting results.

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