A Generalization of a fixed point theorem of HONG-KUN XU

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Abstract: Xu. H. [1] introduced weakly asymptotic contraction and proved that if $T: X \rightarrow X$ is a continuous map where $(X,d)$ is a complete metric space and $\phi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ a map, which is continuous and $\phi(s) < s$ for all $s > 0$, $\phi(0) = 0$ such that given $\epsilon > 0$, there exists $n_\epsilon > 0$ such that $d(T^{[n]} x, T^{[n]} y) \leq \phi(d(x,y)) + \epsilon$, for all $x, y$ in $X$. It is also assumed that some orbit of $T$, i.e., $\{ T^n x : n \in \mathbb{N} \}$ for some $x \in X$ is bounded. Then $T$ has a unique fixed point $y$ in $X$. Also $T^n x \rightarrow y$ as $n \rightarrow \infty$. In this paper, it has been shown that result is still true if the function $\phi$ is assumed to be upper semicontinuous.

Keywords: complete metric space, Cauchy sequence, fixed point, continuous map, upper semicontinuous map, limit superior

Introduction

Let $(X, d)$ be a complete metric space. A map $T: X \rightarrow X$ is said to be contraction map if there exists a constant $c$, $0 < c < 1$ such that $d(Tx, Ty) \leq c d(x, y)$ for all $x, y$ in $X$. Banach proved that in this case, $T$ has a unique fixed point. Due to its wide applications, the theorem has been extended in a number of ways; see, for example, [3],[4] and [5].

In this direction, Kirk [2] introduced the notion of asymptotic contraction which is an asymptotic version of fixed point theorem by Boyd and Wong [7] and proved the fixed point theorem for this class of mappings.

Definition : Let $(M,d)$ be a metric space. A mapping $T: M \rightarrow M$ is said to be an asymptotic contraction if

$$d(T^n x, T^n y) \leq \phi_n(d(x,y)) \text{ for all } x, y \in M, \text{ where } \phi_n : [0,\infty) \rightarrow [0,\infty) \text{ and } \phi_n \rightarrow \phi \text{ uniformly on the range of } d.$$  

where $\phi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is continuous such that $\phi(s) < s$ for all $s > 0$, $\phi(0) = 0$.

Kirk [2] proved the following theorem:

Theorem: Suppose $(M,d)$ is a complete metric space and suppose $T: M \rightarrow M$ is an asymptotic contraction for which the mappings $\phi_n$ are also continuous. Assume also that some orbit of $T$ is bounded. Then $T$ has a unique fixed point $z \in M$, and moreover the Picard sequence $(T^n(x))$ converges to $z$ for each $x \in M$.

The proof given by Kirk is nonconstructive, it uses ultrapower techniques and thus depends on the axiom of choice. Simple proofs of Kirk theorem has been given in [6] and further generalizations of the theorem have been given in [8], [9] and [10].

Xu. H. [1] generalized the result of kirk by introducing weakly asymptotic contractions.

Definition : A continuous mapping $T$ from a complete metric space to itself is said to be weakly asymptotic contraction if given $\epsilon > 0$, there exists $n_\epsilon > 0$ such that $d(T^{[n]} x, T^{[n]} y) \leq \phi(d(x,y)) + \epsilon$, for all $x, y$ in $X$, where $\phi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a map, which is continuous and $\phi(s) < s$ for all $s > 0$, $\phi(0) = 0$.  

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Xu. H. proved that if $T : X \rightarrow X$ is a weakly asymptotic contraction mapping, where $(X, d)$ is a complete metric space. Also assume that some orbit of $T$ i.e. $\{ T^n x : n \in \mathbb{N} \}$ for some $x \in X$ is bounded. Then $T$ has a unique fixed point $y$ in $X$. Also

$$T^n x \rightarrow y \text{ as } n \rightarrow \infty$$

Main purpose of this paper is to show that result of Xu. H. [1] is still valid if the function $\phi$ is assumed to be upper semicontinuous only.

**Main Theorem**: Let $(X, d)$ be a complete metric space. $T : X \rightarrow X$ be a continuous map. Suppose there exists a map $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ which is upper semicontinuous and $\phi(s) < s$ for all $s > 0$, $\phi(0) = 0$.

If $T$ satisfies the following condition:

Given $\epsilon > 0$, there exists $n_\epsilon > 0$ such that

$$d(T^n x, T^n y) \leq \phi(d(x, y)) + \epsilon$$

Assume that some orbit of $T$ i.e. $\{ T^n x : n \in \mathbb{N} \}$ for some $x \in X$ is bounded. Then $T$ has a unique fixed point $y$ in $X$. Also

$$T^n x \rightarrow y \text{ as } n \rightarrow \infty$$

To prove the theorem we need the following lemma:

**Lemma**: If $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is upper semicontinuous and $\phi(s) < s$ for all $s > 0$, $\phi(0) = 0$.

Define the function $\psi(t) = \max \{ \phi(\tau) : \tau \in [0, t] \}$ then

1) $\psi$ is increasing
2) $\psi(s) < s$ for all $s > 0$
3) $\psi$ is upper semicontinuous.

(Note here that every upper semicontinuous map on a compact set assumes its maximum)

**Proof**: clearly $\psi$ is increasing. Now if $s > 0$ then

$$\psi(s) = \phi(\tau^0) \text{ for some } \tau^0 \in [0, s], \text{ so that } \psi(s) < s.$$

Now we show that $\psi$ is upper semicontinuous:

Let $\epsilon > 0$ be given:

Case 1: $t > t_0 > 0$

$$\psi(t) - \psi(t_0) = \max \{ \phi(\tau) : \tau \in [0, t] \} - \max \{ \phi(\tau) : \tau \in [0, t_0] \} \leq \max \{ \phi(\tau) : \tau \in [t_0, t] \} - \phi(t_0) = \max \{ \phi(\tau) - \phi(t_0) : \tau \in [t_0, t] \}$$

Now since $\phi$ is upper semicontinuous, therefore there exists $\delta > 0$ such that $\phi(t) < \phi(t_0) + \epsilon$ whenever $|t - t_0| < \delta$. Thus $\psi(t) - \psi(t_0) < \epsilon$ whenever $|t - t_0| < \delta$.
Case 2 : $0 < t < t_0$

$$
\psi(t) - \psi(t_0) = \max \{ \phi(\tau) : \tau \in [0, t] \} - \max \{ \phi(\tau) : \tau \in [0, t_0] \}
\leq \max \{ \phi(\tau) : \tau \in [t, t_0] \} - \phi(t_0)
= \max \{ \phi(\tau) - \phi(t_0) : \tau \in [t, t_0] \}
$$

Now since $\phi$ is upper semicontinuous, therefore there exists $\delta > 0$ such that

$$
\phi(t) < \phi(t_0) + \epsilon \quad \text{whenever} \quad |t - t_0| < \delta.
$$

Thus $\psi(t) - \psi(t_0) < \epsilon$ whenever $|t - t_0| < \delta$.

Case 3 : $t_0 = 0, t > t_0$

$$
\psi(t) - \psi(t_0) = \max \{ \phi(\tau) : \tau \in [0, t] \} - \max \{ \phi(\tau) - \phi(t_0) : \tau \in [0, t] \}
$$

Now since $\phi$ is upper semicontinuous at $t_0 = 0$, therefore there exists $\delta > 0$ such that

$$
\phi(t) < \phi(t_0) + \epsilon \quad \text{whenever} \quad t - t_0 < \delta.
$$

Thus $\psi(t) - \psi(t_0) < \epsilon$ whenever $t - t_0 < \delta$.

**Proof of main theorem**:

Put $d_{n,m} = d(T^n x, T^m y)$

Let $d_{n,m} = \lim_{n,m \rightarrow \infty} d_{n,m} = \lim_{k \rightarrow \infty} \sup \{ d_{n,m} : n, m \geq k \} < \infty$

Let $\epsilon > 0$ be given:

Now $d_{n,m} = d(T^{n_n} (T^{n} x), T^{m_n} (T^{m} y)) \leq \phi(d(T^{n_n} x), (T^{m_n} y)) + \epsilon$

$$
\leq \psi(d(T^{n_n} x), (T^{m_n} y)) + \epsilon
$$

Taking limit superior we get:

$$
\lim_{n,m \rightarrow \infty} d_{n,m} \leq \lim_{n,m \rightarrow \infty} \left[ \psi(d(T^{n_n} x), (T^{m_n} y)) + \epsilon \right]
= \lim_{n,m \rightarrow \infty} \psi(d(T^{n_n} x), (T^{m_n} y)) + \epsilon
= \lim_{n,m \rightarrow \infty} \psi(d_{n_n,m_n}) + \epsilon
$$

Claim : $\lim_{n,m \rightarrow \infty} \psi(d_{n_n,m_n}) \leq \psi(\lim_{n,m \rightarrow \infty} d_{n_n,m_n})$

Let $\lim_{n,m \rightarrow \infty} d_{n_n,m_n} = L$

$$
\lim_{k \rightarrow \infty} \sup \{ d_{n_n,m_n} : n, m \geq k \} = L
$$

Now as $\psi$ is upper semicontinuous, this implies that

$$
\lim_{k \rightarrow \infty} \psi(\sup \{ d_{n_n,m_n} : n, m \geq k \}) \leq \psi(L)
$$

Clearly $\sup \{ \psi(d_{n_n,m_n}) : n, m \geq k \} \leq \psi(\sup \{ d_{n_n,m_n} : n, m \geq k \})$

Thus we have

$$
\lim_{k \rightarrow \infty} \sup \{ \psi(d_{n_n,m_n}) : n, m \geq k \} \leq \psi(L)
$$

$$
\Rightarrow \lim_{n,m \rightarrow \infty} \psi(d_{n_n,m_n}) \leq \psi(\lim_{n,m \rightarrow \infty} d_{n_n,m_n}), \text{this proves the claim.}
$$

Thus from eq. (1), we get:
\[
\lim_{n,m \to \infty} d_{n,m} = \psi(\lim_{n,m \to \infty} d_{n-n, m-m}) + \varepsilon
\]
\[
\Rightarrow d \to 0 \leq \psi(d) + \varepsilon
\]
Since \( \varepsilon > 0 \) is arbitrary, therefore
\[
d \to 0
\]
\[
\Rightarrow (T^n(x)) \text{ is Cauchy sequence. Since } X \text{ is complete, } (T^n(x)) \text{ will converge to unique fixed point } y \text{ of } X
\]

References: