Fixed Point Theorems for Multivalued Contractive mappings in b-Metric Space

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Abstract

In this paper, we prove a fixed point theorem for multivalued contractive mappings in b-metric spaces. This results offers a generalization of Swati Agarawal, K. Qureshi and Jyoti Nema theorem in [1]. An example to support our result is presented

Key words: b-Metric space; Contraction; Multivalued mappings; Fixed point.

1 Introduction and preliminaries

In many branches of science, economics, computer science, engineering and the development of nonlinear analysis, the fixed point theory is one of the most important tools. In 1989, Bakhtin [2] introduced the concept of b-metric space. In 1993, Czerwik [6] extended the results of b-metric spaces. Using this idea many researcher presented generalization of the renowned Banach fixed point theorem in the b-metric space. Mehmet Kir [8], Boriceanu [5], Czerwik [6], Bota [4], Pacurar [9] extended the fixed point theorem in b-metric space. Various problems of the convergence of measurable functions with respect to measure, Czerwik [6] first presented a generalization of Banach fixed point theorem in b-metric spaces. The following definitions will be needed in the sequel:

Definition 1.1. [6] Let $X$ be a nonempty set and let $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}^+$ is said to be a b-metric if and only if for all $x, y, z \in X$ the following conditions are satisfied:

1) $d(x, y) = 0$ if and only if $x = y$;
2) $d(x, y) = d(y, x)$;
3) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

Then, the triplet $(X, d, s)$ is called a b-metric space.
It is an obvious fact that a metric space is also a b-metric space with 
\( s = 1 \), but the converse is not generally true. To support this fact, we have the following example.

**Example 1.1.** [5] The set \( l_p(\mathbb{R}) \) (with \( 0 < p < 1 \)), where \( l_p(\mathbb{R}) := \{(x_n) \subset \mathbb{R} : \sum_{n=1}^{\infty} |x_n|^p < \infty \}, \) together with the function \( d : l_p(\mathbb{R}) \times l_p(\mathbb{R}) \to \mathbb{R}, \)
\[
d(x, y) = \left( \sum_{n=1}^{\infty} |x_n - y_n|^p \right)^{\frac{1}{p}} \text{ where } x = (x_n), y = (y_n) \in l_p(\mathbb{R}) \text{ is a b-metric space. By an elementary calculation we obtain that } d(x, z) \leq 2^{\frac{1}{p}}[d(x, y) + d(y, z)].
\]

**Example 1.2.** [5] Let \( X = \{0, 1, 2\} \) and \( d(2, 0) = d(0, 2) = m \geq 2, d(0, 1) = d(1, 2) = d(1, 0) = d(2, 1) = 1 \) and \( d(0, 0) = d(1, 1) = d(2, 2) = 0 \). Then \( d(x, y) \leq \frac{2}{m}[d(x, z) + d(z, y)] \) for all \( x, y, z \in X \). If \( m > 2 \) then the triangle inequality does not hold.

**Example 1.3.** [5] The space \( L_p(0 < p < 1) \), for all real function \( x(t), t \in [0, 1] \) such that \( \int_{0}^{1} |x(t) - y(t)|^p dt \).

**Definition 1.2.** [5] Let \( (X, d) \) be a b-metric space. Then a sequence \( \{x_n\} \) in \( X \) is called a Cauchy sequence if and only if for all \( \epsilon > 0 \) there exist \( N \in \mathbb{N} \) such that for all \( n, m \geq N \in \mathbb{N} \), we have \( d(x_n, x_m) \leq \epsilon \).

**Definition 1.3.** [5] Let \( (X, d) \) be a b-metric space. Then a sequence \( \{x_n\} \) in \( X \) is called convergent sequence if and only if there exists \( x \in X \) such that for all \( \epsilon > 0 \) there exists \( N \in \mathbb{N} \in \mathbb{N} \) such that for all \( n \geq N \in \mathbb{N} \), we have \( d(x_n, x) \leq \epsilon \). In this case we write \( \lim_{n \to \infty} x_n = x \).

**Definition 1.4.** [10] Let \( X \) and \( Y \) be nonempty sets, \( T \) is said to be multivalued mapping from \( X \) to \( Y \) if \( T \) is a function for \( X \) to the power set of \( Y \). We denote a multivalued map:
\[ T : X \to 2^Y. \]

**Definition 1.5.** [10] A point of \( x_0 \in X \) is said to be a fixed point of the multivalued mappings \( T \) if \( x_0 \in Tx_0 \).

**Definition 1.6.** [3] Let \( (X, d) \) be a metric space. A map \( T : X \to X \) is called Contraction if there exists \( 0 \leq \lambda < 1 \) such that \( d(Tx, Ty) \leq \lambda d(x, y) \), for all \( x, y \in X \).

**Definition 1.7.** [3] Let \( (X, d) \) be a metric space. We define the Hausdorff metric on \( CB(X) \) induced by \( d. \) That is \( H(A, B) = \max\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\} \) for all \( A, B \in CB(X) \), where \( CB(X) \) denotes the family of all nonempty closed and bounded subsets of \( X \) and \( d(x, B) = \inf\{d(x, b) : b \in B\}, \) for all \( x \in X \).

**Definition 1.8.** [3] Let \( (X, d) \) be a metric space. A map \( T : X \to CB(X) \) is said to be multivalued contraction if there exists \( 0 \leq \lambda < 1 \) such that \( H(Tx, Ty) \leq \lambda d(x, y) \), for all \( x, y \in X \).
In 2016, Swati Agarawal, K. Qureshi and Jyoti Nema proved the following theorem in [1].

**Theorem 1.4.** Let \((X, d)\) be a complete \(b\)-metric space. Let \(T: X \to X\) be a mappings such that

\[
d(Tx, Ty) \leq a \max\{d(x, Tx), d(y, Ty), d(x, y)\} + b\{d(x, Ty) + d(y, Tx)\}
\]

where \(a, b > 0\) such that \(a + 2bs \leq 1\) for all \(x, y \in X\) and \(s \geq 1\) then \(T\) has a unique fixed point.

In 1996, B.E. Rhoades define the contractive definition as follows:

Let \(F: X \to CB(X).\) For each \(x, y \in X,\)

\[
H(Fx, Fy) \leq \alpha d(x, y) + \beta \max\{d(x, Fx), d(y, Ty)\} + \gamma[d(x, Fy) + d(y, Fy)],
\]

(1)

where \(\alpha, \beta, \gamma \geq 0\) and such that

\[
s := \left(\frac{1 + \alpha + \gamma}{1 - \beta - \gamma}\right)\frac{\alpha + \beta + \gamma}{1 - \gamma} < 1.
\]

In this paper, we study the existence of fixed point on above contractive mapping under \(b\)-metric space. Our results generalization of [1].

2 Main results

**Theorem 2.1.** Let \((X, d)\) be a complete \(b\)-metric space with constant \(s \geq 1\) and the mapping \(T: X \to CB(X)\) be multivalued map satisfying

\[
H(Tx, Ty) \leq ad(x, y) + b \max\{d(x, Tx), d(y, Ty)\} + c[d(x, Ty) + d(y, Tx)]
\]

(2)

for all \(x, y \in X,\) and \(a, b, c \in [0, 1)\) are constant such that \(a + b + c < 1.\) Then \(T\) has a unique fixed point in \(X.\)

**Proof.** For every \(x_0 \in X\) and \(n \geq 1, x_1 \in Tx_0\) and \(x_{n+1} \in Tx_n,\) we have

\[
d(x_{n+1}, x_n) \leq H(Tx_n, Tx_{n-1})
\]

\[
\leq ad(x_n, x_{n-1}) + b \max\{d(x_n, Tx_n), d(x_{n-1}, Tx_{n-1})\}
\]

\[
+ c[d(x_n, Tx_{n-1}) + d(x_{n-1}, Tx_{n-1})]
\]

\[
\leq ad(x_n, x_{n-1}) + b \max\{d(x_n, x_{n+1}), d(x_{n-1}, x_n)\}
\]

\[
+ c[d(x_n, x_n) + d(x_{n-1}, x_n)]
\]

\[
\leq (a + c)d(x_n, x_{n-1}) + b \max\{d(x_n, x_{n+1}), d(x_{n-1}, x_n)\}
\]

\[
\leq (a + c)d(x_n, x_{n-1}) + bM_1
\]
where $M_1 = \max \{d(x_n, x_{n+1}), d(x_{n-1}, x_{n})\}$

Now two cases arises,

**Case I:** If suppose that $M_1 = d(x_n, x_{n+1})$ then we have,

\[
d(x_n, x_{n+1}) \leq (a + c)d(x_n, x_{n-1}) + bd(x_n, x_{n+1})
\]
\[
\leq (\frac{a + c}{1 - b})d(x_n, x_{n-1})
\]
\[
\leq kd(x_n, x_{n-1})
\]

where $k = (\frac{a + c}{1 - b}) < 1$

\[
\therefore d(x_n, x_{n+1}) \leq kd(x_{n-1}, x_n),
\]
\[
d(x_n, x_{n+1}) \leq k^2d(x_{n-2}, x_{n-1})
\]

Continuing this process, we obtain

\[
d(x_n, x_{n+1}) \leq k^n d(x_0, x_1).
\]

**Case II:** If suppose that $M_1 = d(x_{n-1}, x_n)$

\[
d(x_n, x_{n+1}) \leq (a + c)d(x_n, x_{n-1}) + bd(x_{n-1}, x_n)
\]
\[
\leq (a + b + c)d(x_n, x_{n-1})
\]
\[
\leq kd(x_n, x_{n-1})
\]

where $k = a + b + c < 1$

\[
\therefore d(x_n, x_{n+1}) \leq kd(x_{n-1}, x_n),
\]
\[
d(x_n, x_{n+1}) \leq k^2d(x_{n-2}, x_{n-1})
\]

Continuing this process, we obtain

\[
d(x_n, x_{n+1}) \leq k^n d(x_0, x_1).
\]

Let $m, n \in N, m > n,$

\[
d(x_n, x_m) \leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_m)],
\]
\[
\leq sd(x_n, x_{n+1}) + s^2[d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_m)],
\]
\[
\leq sd(x_n, x_{n+1}) + s^2d(x_{n+1}, x_{n+2}) + s^3d(x_{n+2}, x_m) + \ldots \ldots
\]
\[
\leq sk^n d(x_0, x_1) + s^2k^{n+1}d(x_{n+1}, x_{n+2}) + s^3k^{n+2}d(x_{n+2}, x_m) + \ldots \ldots
\]
\[
\leq sk^n d(x_0, x_1)[1 + sk + (sk)^2 + (sk)^3 + \ldots \ldots]
\]
\[
\leq \frac{sk^n}{1 - sk} d(x_0, x_1).
\]

Since $k < 1,$ \( \lim_{n \to \infty} \frac{sk^n}{1 - sk} d(x_0, x_1) = 0 \) as $n, m \to \infty$. Then \{\( x_n \)\} is Cauchy sequence in $(X, d)$. Since $(X, d)$ is a complete $b$-metric space, there exists $z \in X$
such that \( x_n \to z \) as \( n \to \infty \).

Now,

\[
d(z, Tz) \leq s[d(z, x_{n+1}) + d(x_{n+1}, Tz)]
\]

\[
\leq sd(z, Tx_n) + sH(Tx_n, Tz)
\]

\[
\leq sd(z, x_{n+1}) + s[ad(x_n, z) + b \max\{d(x_n, Tx_n), d(z, Tz)\}
+ c(d(x_n, Tz) + d(z, Tz))]
\]

\[
\leq sd(z, x_{n+1}) + s[ad(x_n, z) + b \max\{d(x_n, x_{n+1}), d(z, Tz)\}
+ c(d(x_n, Tz) + d(z, Tz))]
\]

\[
\leq sd(z, x_{n+1}) + sad(x_n, z) + sb \max\{d(x_n, x_{n+1}), d(z, Tz)\}
+ sc(d(x_n, Tz) + d(z, Tz))
\]

\[
\leq sd(z, x_{n+1}) + sad(x_n, z) + sbM_2 + sc(d(x_n, Tz) + d(z, Tz))
\]

where \( M_2 = \max\{d(x_n, x_{n+1}), d(z, Tz)\} \).

**Case I:** Suppose that \( M_2 = d(x_n, x_{n+1}) \) then we have,

\[
d(z, Tz) \leq sd(z, x_{n+1}) + sad(x_n, z) + sbd(x_n, x_{n+1}) + sc(d(x_n, Tz) + d(z, Tz))
\]

\[
\leq sd(z, x_{n+1}) + sad(x_n, z) + s^2bd(x_n, z) + s^2bd(z, x_{n+1})
+ s^2cd(x_n, z) + s^2cd(z, Tz) + scd(z, Tz)
\]

\[
\leq (\frac{s + s^2b}{1 - s^2c - sc})d(z, x_{n+1}) + (\frac{sa + s^2b + s^2c}{1 - s^2c - sc})d(x_n, z)
\]

Letting \( n \to \infty \), we get \( d(z, Tz) = 0 \).

Thus, \( Tz = z \). Therefore \( z \) is the fixed point of \( T \).

**Case II:** Suppose that \( M_2 = d(z, Tz) \) then we have,

\[
d(z, Tz) \leq sd(z, x_{n+1}) + sad(x_n, z) + sd(z, Tz) + scd(x_n, Tz) + d(z, Tz)
\]

\[
\leq sd(z, x_{n+1}) + sad(x_n, z) + sd(z, Tz)
+ s^2cd(x_n, z) + s^2cd(z, Tz) + scd(z, Tz)
\]

\[
\leq (\frac{s}{1 - s - s^2c - sc})d(z, x_{n+1}) + (\frac{sa + s^2b}{1 - s^2c - sc})d(x_n, z)
\]

Letting \( n \to \infty \), we get \( d(z, Tz) = 0 \).

Thus, \( Tz = z \). Therefore \( z \) is the fixed point of \( T \). Now we show that \( z \) is the unique fixed point of \( T \). Assume that \( u \) is another fixed point of \( T \). Then we have \( Tv = v \) and

\[
d(u, v) = d(Tz, Tv)
\]

\[
\leq s[d(u, Tv) + d(v, Tu)]
\]

we obtain, \( d(u, v) \leq 2sd(u, v) \). This implies that \( u = v \). Therefore \( T \) has a fixed point in \( X \).
Example 2.2. Let $X = [0, 1]$. We define $d: X \times X \to X$ by $d(x, y) = |x - y|^2$ for all $x, y \in X$. Then $(X, d)$ is a complete $b$-metric space. Define $T: X \to \text{CB}(X)$ by $Tx = \frac{x}{2}$ for all $x, y \in X$. Then,

$$H(Tx, Ty) = \frac{1}{49} d(x, y)$$

[where, $b = c = 0, a = \frac{1}{49}$]. Therefore, $0 \in X$ is the unique fixed point of $T$.

References


