Some generalized expectations associated with probability density functions of various multivariable distributions II

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ABSTRACT

In this paper, we employ the multivariable I-function defined by Prasad [3] to derive various generalized expectations associated with probability density functions of various multivariate distributions (beta, gamma, Dirichlet distributions).

KEYWORDS: I-function of several variables, generalized expectations, probability density functions, multivariate distributions.

2010 Mathematics Subject Classification. 33C60, 82C31

1. Introduction

Recently Chandel and Gupta [2] are employed the multivariable H-function defined by Srivastava and Panda [4] in deriving expectations associated with probability density functions of various multivariate distributions. Here in the present document, we extend the work with the multivariable I-function defined by Prasad [3]. The I-function of several variables generalize the multivariable H-function defined by Srivastava et al [4], itself is a generalization of G-function of several variables. The multivariable I-function is defined in term of multiple Mellin-Barnes type integral:

\[ I(z_1, z_2, \ldots, z_r) = \int_{p_2,q_2,p_3,q_3,\ldots}^{n_1,n_2,\ldots,n_r} (a_{2j}; \alpha'_{2j}, \alpha''_{2j})_{1,p_2}^{(r)} \cdots \left( b_{2j}; \beta'_{2j}, \beta''_{2j} \right)_{1,q_2}^{(r)} \left( z_1 \right)_{(a_{2j}; \alpha_{2j}, \alpha''_{2j})_{1,p_2}^{(r)}} \cdots \left( z_r \right)_{(b_{2j}; \beta_{2j})_{1,q_2}^{(r)}} \]

\[ = \frac{1}{(2\pi i)^r} \int_{L_1} \cdots \int_{L_r} \xi(t_1, \ldots, t_r) \prod_{i=1}^{s} \phi_i(s_i) z_i^{s_i} ds_1 \cdots ds_r \]

The defined integral of the above function, the existence and convergence conditions, see Y.N Prasad [3]. Throughout the present document, we assume that the existence and convergence conditions of the multivariable I-function.

The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as:

\[ |\arg z_k| < \frac{1}{2} \Omega_i^{(k)} \pi, \text{ where} \]

ISSN: 2231-5373 http://www.ijmttjournal.org Page 84
The complex numbers $z_i$ are not zero. Throughout this document, we assume the existence and absolute convergence conditions of the multivariable I-function.

We may establish the asymptotic expansion in the following convenient form:

\[ I(z_1, \cdots, z_r) = 0( |z_1|^{\alpha'_1}, \cdots, |z_r|^{\alpha'_r} ) \rightarrow 0 \]
\[ I(z_1, \cdots, z_r) = 0( |z_1|^{\beta'_1}, \cdots, |z_r|^{\beta'_r} ) \rightarrow \infty \]

where $k = 1, \cdots, z : \alpha'_k = \min \{ \text{Re}(b_j^{(k)}) / \alpha_j^{(k)} \}, j = 1, \cdots, m_k$ and
\[ \beta'_k = \max \{ \text{Re}(\alpha_j^{(k)} - 1) / \alpha_j^{(k)} \}, j = 1, \cdots, n_k \]

We will use these following notations in this paper:

\[ U = p_2, q_2, q_3, \cdots, p_{r-1}, q_{r-1} \]
\[ V = 0, n_2; 0, n_3; \cdots; 0, n_{s-1} \]

\[ W = (p', q'); \cdots; (p^{(r)}, q^{(r)}); X = (m', n'); \cdots; (m^{(r)}, n^{(r)}) \]

\[ A = (a_{2k}, \alpha'_{2k}, \alpha''_{2k}); \cdots; (a_{(r-1)k}, \alpha'_{(r-1)k}, \alpha''_{(r-1)k}); \alpha_{(r-1)k} \]

\[ B = (b_{2k}, \beta'_{2k}, \beta''_{2k}); \cdots; (b_{(r-1)k}, \beta'_{(r-1)k}, \beta''_{(r-1)k}); \beta_{(r-1)k} \]

\[ \mathfrak{a} = (a_{rk}; \alpha'_{rk}, \alpha''_{rk}; \cdots, \alpha_{rk}^{(r)}); \mathfrak{b} = (b_{rk}; \beta'_{rk}, \beta''_{rk}; \cdots, \beta_{rk}^{(r)}) \]

\[ A' = (a_k, \alpha_k)^{1, p'}; \cdots; (a_k, \alpha_k^{(r)})^{1, p^{(r)}}; B' = (b_k, \beta_k)^{1, p'}; \cdots; (b_k, \beta_k^{(r)})^{1, p^{(r)}} \]

The multivariable I-function write:

\[ I(z_1, \cdots, z_r) = I_{U \cup P_{r}, q_{r}}^{V, 0, n_{s}; X} \left( \begin{array}{c|c|c}
  z_1 & A ; \mathfrak{a} ; A' \\
  \vdots & \mathfrak{b} ; B ; B' \\
  z_r & B ; \mathfrak{b} ; B'
\end{array} \right) \]

2. Required results
3. Multivariate Gamma distribution

\[ f(x_1, \cdots, x_r) = \frac{\Gamma(\mu_1) \cdots \Gamma(\mu_r) \lambda^{\mu_1+\cdots+\mu_r}}{\Gamma(\mu+\mu_1+\cdots+\mu_r)} e^{-(x_1+\cdots+x_r)} (x_1+\cdots+x_r)^{\mu-1} x_1^{\mu_1-1} \cdots x_r^{\mu_r-1} \]  

where \( Re(\lambda) > 0, Re(\mu) > 0, Re(\mu_i) > 0, i = 1, \cdots, r \) and \( f(x_1, \cdots, x_r) = 0 \) elsewhere.

With the help of the equations (2.1) and (2.2), we obtain the following result.

\[ \int_0^\infty \cdots \int_0^\infty f(x_1 + \cdots + x_r) \, dx_1 \cdots dx_r = 1 \]  

Therefore \( f(x_1, \cdots, x_r) \) is a probability density function for multivariate gamma distribution.

4. Expectations associated with multivariate gamma distribution

Corresponding to density function \( f(x_1, \cdots, x_r) \) defined in (3.1), the expectation value of the function \( g(x_1, \cdots, x_r) \) is defined as

\[ \langle g(x_1, \cdots, x_r) \rangle = \int_0^\infty \cdots \int_0^\infty f(x_1, \cdots, x_r) g(x_1, \cdots, x_r) \, dx_1 \cdots dx_r \]  

Consider \( g_1(x_1, \cdots, x_r) = I_{U_{Pr,qr};W}^{\gamma_1,\cdots,\gamma_r,x} \)

\[
\begin{pmatrix}
  z_1 x_1^{\gamma_1} & \cdots & x_r^{\gamma_1} (x_1 + \cdots + x_r)^{\nu_1} \\
  \vdots & \ddots & \vdots \\
  z_r x_1^{\gamma_r} & \cdots & x_r^{\gamma_r} (x_1 + \cdots + x_r)^{\nu_r} \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
  A \\
  \vdots \\
  B \\
\end{pmatrix}
\]
where $I_{U;pr,q_r;W}^{V;0,n_r;X}$ is the I-function of $r$ variables.

With the help of (4.1), (2.1) and (2.2), the expectation of $g_1(x_1, \ldots, x_r)$ corresponding to density function $f(x_1, \ldots, x_r)$ defined by (3.1), is given by

$$
\langle g_1(x_1, \ldots, x_r) \rangle = \frac{\Gamma(\mu_1) \cdots \Gamma(\mu_r) \lambda^{\mu_1 + \mu_1 + \cdots + \mu_r}}{\Gamma(\mu + \mu_1 + \cdots + \mu_r)} \int_0^\infty x_1^{\mu_1-1} \cdots x_r^{\mu_r-1} 1_{U;pr,q_r;W}^{V;0,n_r;X} \int_0^\infty \cdots \int_0^\infty e^{-(x_1 + \cdots + x_r)} (x_1 + \cdots + x_r)^\mu \ dx_1 \cdots dx_r
$$

$$
= \frac{\Gamma(\mu_1) \cdots \Gamma(\mu_r)}{\Gamma(\mu + \mu_1 + \cdots + \mu_r)} \int_{U;pr,\alpha_r;W}^{V;0,n_r+r+1;X} \begin{vmatrix}
\alpha_1^{v_0} & \cdots & \alpha_r^{v_0} \\
\vdots & \ddots & \vdots \\
\alpha_r^{v_r} & \cdots & \alpha_r^{v_r}
\end{vmatrix} \begin{pmatrix}
1 - \mu_1 - \cdots - \mu_r; v_1 + \alpha_1^{v_1} + \cdots + \alpha_r^{v_r} + \alpha_1^{v_r} + \cdots + \alpha_r^{v_r}
\vdots \\
\alpha_1^{v_r} + \cdots + \alpha_r^{v_r} + \alpha_1^{v_r} + \cdots + \alpha_r^{v_r}
\end{pmatrix} \left[ 1 - \frac{1}{\mu_j} : \alpha_j^{v_j}, \cdots, \alpha_r^{v_r} \right]_{1,r}
$$

(4.3)

Provided that

a) $Re(\lambda) > 0$, $Re(\mu_i) > 0$, $Re(v_i) > 0$, $Re(\alpha_j^{v_j}) > 0$, $x_i \geq 0$, $i = 1, \cdots, r$

b) $\left| \frac{arg z_k}{\lambda^{v_1} + \alpha_1^{v_1} + \cdots + \alpha_r^{v_r}} \right| < \frac{1}{2} \Omega_1^{(k)} \pi$, where $\Omega_1^{(k)}$ is given in (1.3)

c) $f(x_1, \ldots, x_r) = 0$ elsewhere.

5. Multivariate Beta distribution

Consider the function

$$
F(x_1, \ldots, x_r) = \frac{\Gamma(\mu_1 + \cdots + \mu_r) \Gamma(\alpha + \beta + \mu_1 + \cdots + \mu_r)}{\Gamma(\mu_1) \cdots \Gamma(\mu_r) \Gamma(\beta) \Gamma(\alpha + \mu_1 + \cdots + \mu_r)} (1 + x_1 + \cdots + x_r)^{\alpha + \beta + \mu_1 + \cdots + \mu_r}
$$

(5.1)
where $Re(\alpha) > 0$, $Re(\beta) > 0$, $Re(\mu_i) > 0$, $i = 1, \cdots, r$ and $F(x_1, \cdots, x_r) = 0$ elsewhere. Now use to (2.1) and (2.3), we have
\[
\int_0^\infty \cdots \int_0^\infty F(x_1, \cdots, x_r) \, dx_1 \cdots dx_r = 1 \tag{5.2}
\]

6. Expectation associated with multivariate Beta distribution

Corresponding to probability density function $F(x_1, \cdots, x_r)$ defined by (5.1) consider the function

\[
G(x_1, \cdots, x_r) = G_1(x_1, \cdots, x_r) = I_{U;\, p_r, q_r; \, W}^{V;\, 0, n_r; \, X} \begin{pmatrix}
\frac{x_1^{\alpha_1} \cdots x_r^{\alpha_r} (x_1 + \cdots + x_r)^{n_1}}{(1 + x_1 + \cdots + x_r)^{\alpha_1 + \cdots + \alpha_r + n_1}} \\
\vdots \\
\frac{x_r^{\alpha_1} \cdots x_r^{\alpha_r} (x_1 + \cdots + x_r)^{n_r}}{(1 + x_1 + \cdots + x_r)^{\alpha_1 + \cdots + \alpha_r + n_r}}
\end{pmatrix}
\begin{pmatrix}
A ; \, A' \\
\vdots \\
B ; \, B'
\end{pmatrix}
\]

(6.1)

Therefore expectation of $G_1(x_1, \cdots, x_r)$ is given by

\[
\langle G_1(x_1, \cdots, x_r) \rangle = \frac{\Gamma(\mu_1 + \cdots + \mu_r) \Gamma(\alpha + \beta + \mu_1 + \cdots + \mu_r)}{\Gamma(\mu_1) \cdots \Gamma(\mu_r) \Gamma(\beta) \Gamma(\alpha + \mu_1 + \cdots + \mu_r)} (x_1 + \cdots + x_r)\alpha
\]

\[
\int_0^\infty \cdots \int_0^\infty \frac{(x_1 + \cdots + x_r)^{\alpha_1 - 1} \cdots x_r^{\mu_r - 1}}{(1 + x_1 + \cdots + x_r)^{\alpha_r + \mu_1 + \cdots + \mu_r}} I_{U;\, p_r, q_r; \, W}^{V;\, 0, n_r; \, X} \begin{pmatrix}
\frac{x_1^{\alpha_1} \cdots x_r^{\alpha_r} (x_1 + \cdots + x_r)^{n_1}}{(1 + x_1 + \cdots + x_r)^{\alpha_1 + \cdots + \alpha_r + n_1}} \\
\vdots \\
\frac{x_r^{\alpha_1} \cdots x_r^{\alpha_r} (x_1 + \cdots + x_r)^{n_r}}{(1 + x_1 + \cdots + x_r)^{\alpha_1 + \cdots + \alpha_r + n_r}}
\end{pmatrix}
\begin{pmatrix}
A ; \, A' \\
\vdots \\
B ; \, B'
\end{pmatrix}
\]

\[dx_1 \cdots dx_r = \frac{\Gamma(\mu_1 + \cdots + \mu_r) \Gamma(\alpha + \beta + \mu_1 + \cdots + \mu_r)}{\Gamma(\mu_1) \cdots \Gamma(\mu_r) \Gamma(\alpha + \mu_1 + \cdots + \mu_r)} (x_1 + \cdots + x_r)\alpha
\]

\[
I_{U;\, p_r, q_r; \, W}^{V;\, 0, n_r + r + 1; \, X} \begin{pmatrix}
z_1 \\
\vdots \\
z_r
\end{pmatrix}
\begin{pmatrix}
A ; \, [1, \mu_j : \alpha_j^1, \cdots, \alpha_j^\gamma_{1,r}] \\
\vdots \\
B ; \, (1 - \mu_1 - \cdots - \mu_r ; \alpha_1^1 + \cdots + \alpha_1^\gamma, \cdots, \alpha_r^1 + \cdots + \alpha_r^\gamma)
\end{pmatrix}
\]
\[ (1 - \alpha - \mu_1 - \cdots - \mu_r; \eta_1 + \alpha_1 + \cdots + \alpha_r, \cdots, \eta_r + \alpha_1 + \cdots + \alpha_r, 2; A') \]
\[ (1 - \beta - \alpha - \mu_1 - \cdots - \mu_r; \eta_1 + \alpha_1 + \cdots + \alpha_r, \cdots, \eta_r + \alpha_1 + \cdots + \alpha_r, B; B') \]  
(6.2)

Provided that

a) \( \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\mu_i) > 0, \Re(\nu_i) > 0, \Re(\alpha) > 0, x_i \geq 0, i = 1, \ldots, r \)

b) \( |arg z_k| < \frac{1}{2} \Omega_i^{(k)} \pi, \) where \( \Omega_i^{(k)} \) is given in (1.3)

c) \( f(x_1, \cdots, x_r) = 0 \) elsewhere, \( \alpha_i, \eta_i (i, j = 1, \cdots, r) \) are non-negative real numbers

8. Multivariate Dirichlet distribution

Consider the function defined by

\[ H(x_1, \cdots, x_r) = \frac{\Gamma(\mu_1 + \cdots + \mu_r) \Gamma(\alpha + \beta + \mu_1 + \cdots + \mu_r)(x_1 + \cdots + x_r)^{\alpha + \mu_1 + \cdots + \mu_r}}{\Gamma(\mu_1) \cdots \Gamma(\mu_r) \Gamma(\beta) \Gamma(\alpha + \mu_1 + \cdots + \mu_r)} x_1^{\mu_1 - 1} \cdots x_r^{\mu_r - 1} \]  
(8.1)

at any point in the domain \( x_i \geq 0, x_1 + \cdots + x_r \leq 1, \Re(\mu_i) > 0, \Re(\alpha) > 0, \Re(\beta) > 0, i = 1, \cdots, r \)

and \( H(x_1, \cdots, x_r) = 0 \) elsewhere.

Therefore

\[ \int_0^\infty \cdots \int_0^\infty H(x_1, \cdots, x_r)dx_1 \cdots dx_r = \frac{\Gamma(\mu_1 + \cdots + \mu_r) \Gamma(\alpha + \beta + \mu_1 + \cdots + \mu_r)}{\Gamma(\mu_1) \cdots \Gamma(\mu_r) \Gamma(\beta) \Gamma(\alpha + \mu_1 + \cdots + \mu_r)} \]

\[ \int_0^\infty \cdots \int_0^\infty x_1^{\mu_1 - 1} \cdots x_r^{\mu_r - 1}(x_1 + \cdots + x_r)^{\alpha + \mu_1 + \cdots + \mu_r}[1 - (x_1 + \cdots + x_r)]^{\beta - 1}dx_1 \cdots dx_r \]

\[ = \frac{\Gamma(\alpha + \beta + \mu_1 + \cdots + \mu_r)}{\Gamma(\alpha + \mu_1 + \cdots + \mu_r) \Gamma(\beta)} \int_0^\infty t^{\alpha + \mu_1 + \cdots + \mu_r}(1 - t)^{\beta - 1}dt = 1 \]  
(8.2)

Hence \( H(x_1, \cdots, x_r) \) is probability density function for multivariate Dirichlet distribution.

9. Expectation associated with multivariate Dirichlet distribution

Corresponding to density function \( H(x_1, \cdots, x_r) \) defined in (8.1), the expectation value of the function

is defined as \( h(x_1, \cdots, x_r) \)

\[ \langle h(x_1, \cdots, x_r) \rangle = \int_0^\infty \cdots \int_0^\infty h(x_1, \cdots, x_r)H(x_1, \cdots, x_r)dx_1 \cdots dx_r \]  
(9.1)

Further consider \( h_1(x_1, \cdots, x_r) \) defined by
Thus corresponding to probability density function $H(x_1, \cdots, x_r)$ defined by (8.1), the expectation value of $h_1(x_1, \cdots, x_r)$ is given by

$$
\langle h_1(x_1, \cdots, x_r) \rangle = \frac{\Gamma(\mu_1 + \cdots + \mu_r) \Gamma(\alpha + \beta + \mu_1 + \cdots + \mu_r)}{\Gamma(\mu_1) \cdots \Gamma(\mu_r) \Gamma(\beta) \Gamma(\alpha + \mu_1 + \cdots + \mu_r)} I_{U;\beta}^{V_0,n_r+r+2,X} \left( \begin{array}{c} z_1 \\ \vdots \\ A \end{array} ; \begin{array}{c} z_r \\ A^r \end{array} \right)
$$

(9.3)

Provided that

a) $Re(\alpha) > 0, Re(\beta) > 0, Re(\mu_i) > 0, i = 1, \cdots, r$

b) $|arg z_k| < \frac{1}{2} \Omega_i^{(k)} \pi$, where $\Omega_i^{(k)}$ is given in (1.3)

c) $F(x_1, \cdots, x_r) = 0$ elsewhere, $\alpha_i^2, \gamma_i (i, j = 1, \cdots, r)$ are non-negative real numbers

Remark

If $U = V = A = B = 0$, the multivariable I-function defined by Prasad degenerates in multivariable H-function defined by Srivastava et al [4], for more details, see Chandel and Gupta [2].

10. Conclusion

In this paper we have evaluated the expectation concerning three various multivariate distributions involving the multivariable I-function defined by Prasad [3]. The formulas established in this paper is of very general nature as it contains multivariable H-function, which is a general function of several variables studied so far. Thus, the integral established in this research work would serve as a key formula from which, upon specializing the parameters, as many as desired results involving the special functions of one and several variables can be obtained.

REFERENCES

