On Embedding of Every Finite Group into a Group of Automorphism

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Abstract—In this Article, We have proved that every group of finite order can be embedded into a group of automorphism. We have used the famous classic result of Cayley, which states that every group can be embedded into a group of permutations.

Keywords—Embedding, Finite Group, Group of Automorphism, Aut(Z\times Z\times Z\times \cdots n copies)

Notation:

\text{Aut}(\mathbb{Z}^n) = \text{Aut}(\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \cdots n \text{ copies}) is

group of automorphisms with respect to composition of function.

GL(n,\mathbb{Z}) = \{ [a_{ij}]_{n \times n} : a_{ij} \in \mathbb{Z} \text{ & det}([a_{ij}]_{n \times n}) = \pm 1 \}

Theorem 1: (Cayley’s) Every finite group can be embedded in \( S_n \) for some \( n \in N \).

Theorem 2: GL(n,\mathbb{Z}) is isomorphic to \text{Aut}(\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \cdots n \text{ copies}).

Theorem 3: \( S_n \) can be embedded in GL(n,\mathbb{Z}) for all \( n \in N \).

Main Theorem: Using theorem 1, 2 & 3 we can say that every finite group can be embedded into group \text{Aut}(\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \cdots n \text{ copies}) for some \( n \in N \).

Proof of Theorem 2:

Let \( f, g \in \text{Aut}(\mathbb{Z}^n) \)

Then

\[
\begin{align*}
  f(1,0,0,0\ldots) &= (a_{11}, a_{12}, \ldots, a_{1n}) \\
  f(0,1,0,0\ldots) &= (a_{21}, a_{22}, \ldots, a_{2n}) \\
  &\quad \vdots \\
  f(0,0,0,\ldots,1) &= (a_{n1}, a_{n2}, \ldots, a_{nn})
\end{align*}
\]

Now define \( \varphi : \text{Aut}(\mathbb{Z}^n) \longrightarrow GL(n,\mathbb{Z}) \)

Such that

\[
\varphi(f) = \begin{pmatrix}
  a_{11} & \cdots & a_{1n} \\
  \vdots & \ddots & \vdots \\
  a_{n1} & \cdots & a_{nn}
\end{pmatrix}
\]

\( \Rightarrow f \) is associated with a \( n \times n \) matrix with determinant \( \pm 1 \).

If \( f \) is an automorphism then \( f \) has an inverse automorphism which is multiplication by

\[
N = \begin{pmatrix}
  b_{11} & \cdots & b_{1n} \\
  \vdots & \ddots & \vdots \\
  b_{n1} & \cdots & b_{nn}
\end{pmatrix}
\]

Such that \( MN = I \).

The determinants are integers satisfying \( \det(M) \cdot \det(M^{-1}) = 1 \)

\( \Rightarrow \det(M) = \pm 1 \)

\( \Rightarrow \) There is a bijection between \( GL(n,\mathbb{Z}) \) and \( \text{Aut}(\mathbb{Z}^n) \).

Composition of automorphisms corresponds to multiplication of matrices.

So, it is an isomorphism.

\( \Rightarrow \text{Aut}(\mathbb{Z}^n) \) is isomorphic to \( GL(n,\mathbb{Z}) \).

Proof of Theorem 3:

Let \( S_n \) be the permutation group on \( n \) symbols.

Define \( \varphi : S_n \longrightarrow GL(n,\mathbb{Z}) \) such that:

\[
\varphi(\sigma) = [\sigma]_{n \times n} \quad \forall \sigma \in S_n
\]

Where \([\sigma]_{n \times n}\) is permutation matrix obtained by \( \sigma \).

i.e. if \( \sigma = \begin{pmatrix}
  1 & 2 & \cdots & n \\
  \beta_1 & \beta_2 & \cdots & \beta_n
\end{pmatrix} \) then

\[
[\sigma]_{n \times n} = \begin{pmatrix}
  R_1 \\
  R_2 \\
  \vdots \\
  R_n
\end{pmatrix}
\]

Where \( R_i \) is \( R_i^{th} \) row of identity matrix.
Clearly $\varphi$ is a homomorphism. Now consider the kernel of this homomorphism.

$$\ker \varphi \equiv \{ \sigma : \varphi(\sigma) = I_{\text{inv}} \}$$

$\implies i = \beta_i \quad \forall i$

$\implies \ker \varphi$ is trivial.

Hence the homomorphism is injective.

$\implies S_n$ can be embedded in $GL(n, \mathbb{Z})$ for all $n \in \mathbb{N}$.

**Proof of Main Theorem:**

Since every finite group can be embedded in $S_n$ as stated in theorem 1 & by theorem 3 we can say that $S_n$ can be embedded in $GL(n, \mathbb{Z})$.

So, every finite group can be embedded in $GL(n, \mathbb{Z})$.

And in theorem 2, we have proved that $GL(n, \mathbb{Z})$ is isomorphic to $Aut(\mathbb{Z}^n)$.

Now, combining all these statements one can easily conclude that every finite group is isomorphic to a subgroup of a group of automorphisms.

**Conclusion**

Every finite group is isomorphic to a subgroup of group of permutations by the famous Cayley’s theorem.

But we have proved that every finite group is isomorphic to the subgroup of a group of automorphisms i.e. $Aut(\mathbb{Z}^n)$.

**References**
