Exact Solutions of Some Nonlinear Partial Differential Equations via Extended \( \left( \frac{G'}{G} \right) \)-Expansion Method

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Abstract: In this paper, we apply the extended \( \left( \frac{G'}{G} \right) \)-expansion method for solving the Burger’s equation, the Korteweg-de Vries-Burgers (KdV) equation and the Lax’ fifth-order (Lax5) equation. With the aid of mathematical software Maple, some exact solutions for these equations are successfully.

Keywords: The extended \( \left( \frac{G'}{G} \right) \)-expansion method, exact solutions, some nonlinear partial differential equations.

1. Introduction

Mathematical modeling of many real phenomena leads to non-linear ordinary or partial differential equations in various fields of physics and engineering. There are some methods to obtain approximate or exact solutions of these kinds of equations, such as: the extended tanh-function method [1-4], the sub-equation method [5, 6], the Bäcklund transform method [7], the Exp-function method [8-17], the simple equation method [18-27], the modified extended tanh and fractional Riccati equation [26-32], the Fractional sub-equation method [33-36], the sine-cosine method [37-39], the \( \left( \frac{G'}{G} \right) \)-expansion method [40-44], the \( \left( \frac{G'}{G} \right) \)-expansion method [45-47], the modified simple equation method [48-50], the Kudryashov method [51-53], and so on.

In this paper we have considered the following NPDEs

(I) Burger’s equation

\[ u_t - u_{xx} - au_x = 0. \]  

(II) Generalized Burgers-KdV Equation

\[ u_t + pu^{m}u_x + qu_{xx} - ru_{xxx} = 0. \]

(III) Lax’ Fifth-Order (Lax5) Equation

\[ u_t + u_{xxxx} + 10uu_{xx} + 20u_xu_{xx} + 30u^2u_x = 0. \]

This paper is arranged as follows: In Section 2, we give the description for main steps of the extended \( \left( \frac{G'}{G} \right) \)-expansion method. In Section 3, we apply this method to finding exact solutions for the equations which we stated above.

2. Description Of Extended \( \left( \frac{G'}{G} \right) \)-Expansion Method

Consider the following nonlinear evolution equation, say in the two independent variables \( x, t \)

\[ P(u, D_t u, D_x u, D_x^2 u, D_x^3 u, \ldots) = 0. \]  

Where \( P \) is a polynomial in \( u(x, t) \) and its partial derivatives in which the highest order derivatives and nonlinear terms are involved. In the following, we give the main steps of this method: [40-44]

Step1. Using the wave transformation

\[ u(x, t) = u(\xi), \quad \xi = kx + ct, \]

where \( c \) is a constant to be determined later. Then equation (2.1) becomes a nonlinear ordinary differential equation

\[ Q(u, u', u'', u'''..., \ldots) = 0. \]  

where \( Q \) is a polynomial of \( u \) and its derivatives and the superscripts indicate the ordinary derivatives with respect to \( \xi \). If possible, we should integrate Eq. (2.3) term by term one or more times.

Step2. Suppose the solutions of Eq. (2.3) can be expressed as a polynomial of \( \left( \frac{G'}{G} \right) \) in the form

\[ u(\xi) = \sum_{i=0}^{M} a_i \left( \frac{G'}{G} \right)^i, \]

where \( a_i (i = 0, 1, \ldots, M) \) in Eq. (2.4) are constants to be determined later. The positive integer \( M \) in Eq. (2.4) can be determined by considering the homogeneous balance between the highest-order derivatives and nonlinear terms appearing in Eq. (2.3). More precisely, we define the degree of \( u(\xi) \) as \( D[u(\xi)] = M \) which gives rise to the degree of other expressions as follows:
Therefore can get the value of M in Eq. (2.4).

If M is equal to a fractional or negative number we can take the following transformations: [54]

1- When \( M = \frac{q}{p} \) (where \( M = \frac{q}{p} \) is a fraction in lowest terms), we let

\[
u(\xi) = \nu^M(\xi).
\] (2.6)

Substituting Eq. (2.6) into Eq. (2.3) and then determine the value of M in new Eq. (2.3).

2- When M is a negative integer, we let

\[
u(\xi) = \nu^M(\xi).
\] (2.7)

Substituting Eq. (2.7) into Eq. (2.3) and return to determine the value of M once again.

The function \( G = G(\xi) \) in Eq. (2.4) satisfies the following second order linear ODE:

\[G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0, \] (2.8)

where \( \lambda \) and \( \mu \) are real constants to be determined.

**Step3.** Substituting Eq. (2.4) along with Eq. (2.8), into Eq. (2.3), collecting all terms with the same order of \( \left( \frac{G'}{G} \right) \) together, and then equating each coefficient of the resulting polynomial to zero, we obtain a set of algebraic equations for \( a_i, c, \mu, \lambda, k \).

Then, we solve the system with the aid of a computer algebra system, such as Maple, to determine these constants. On the other hand, depending on the sign of the discriminant({eq}\Omega = \lambda^2 - 4\mu \)\), the general solutions of Eq. (2.8) are as follows:

\[
\left( \frac{G'}{G} \right) = \begin{cases} 
\sqrt{\frac{\lambda}{\mu}} \left( e^{c_1 \sinh \left( \frac{\lambda}{2\mu} \right)} + e^{c_2 \cosh \left( \frac{\lambda}{2\mu} \right)} \right) - \frac{\lambda^2}{2\mu} \Omega > 0, \\
\sqrt{\frac{-\lambda}{\mu}} \left( e^{c_1 \sinh \left( \frac{-\lambda}{2\mu} \right)} + e^{c_2 \cosh \left( \frac{-\lambda}{2\mu} \right)} \right) - \frac{\lambda^2}{2\mu} \Omega < 0, \\
\frac{c_1}{e^{c_2 \cosh \left( \frac{\lambda}{2\mu} \right)} + e^{c_2 \cosh \left( \frac{\lambda}{2\mu} \right)}} - \frac{\lambda^2}{2\mu} \Omega = 0,
\end{cases}
\] (2.9)

where \( c_1, c_2 \) are arbitrary constants. Then substituting \( a_i, c, \mu, \lambda, k \) and along with Eq. (2.9) into Eq. (2.4), we get the solutions of Eq. (2.1).

3. Applications

3.1-Exact solutions of the Burger’s equation

\[
\frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} - au\frac{\partial}{\partial x} = 0.
\] (3.1)

In [55], the author solved Eq. (3.1) by the tanh-coth method and established some exact solutions for it.

Now we will apply the extended \( \left( \frac{G'}{G} \right) \) Expansion Method to Eq. (3.1). To begin with, suppose the \( u(x,t) = \nu(\xi), \xi = x - ct \), where \( c \) is an arbitrary constant to be determined later, to convert the Eq. (3.1) into the following nonlinear (ODE)

\[
c\frac{du}{dt} + \frac{du}{dt^2} + au \frac{du}{dt} = 0, \quad 0 < \alpha < 1. (3.2)
\]

Integrating (3.2) once with respect to \( \xi \) and neglecting the constant of integration, we have

\[
cu + \frac{du}{dt} + au^2 = 0. \quad (3.3)
\]

Balancing \( \left( \frac{du}{dt} \right) \) with \( (u^2) \), we obtain \( (M = 1) \). Thus Eq. (3.3) becomes

\[
u(\xi) = a_{-1} \left( \frac{G'}{G} \right)^{-1} + a_0 + a_1 \left( \frac{G'}{G} \right). \quad (3.4)
\]

Using Eq. (3.4) along with Eq. (2.8), we derive:

\[
\frac{du}{dt} = \mu a_{-1} \left( \frac{G'}{G} \right)^{-1} + \lambda a_{-1} \left( \frac{G'}{G} \right) + (a_{-1} - \mu a_1) \left( \frac{G'}{G} \right)^0 - \lambda a_1 \left( \frac{G'}{G} \right)^1 - a_1 \left( \frac{G'}{G} \right)^2.
\] (3.5)

Substituting Eq. (3.4) and Eq. (3.5) into Eq. (3.3), collecting the coefficients of powers of \( \left( \frac{G'}{G} \right) \) and setting them to zero, we obtain the following system of algebraic equations involving the parameters \( a_i, i = 0, 1, \lambda, \mu, k \) as follows:

\[
\left( \frac{G'}{G} \right)^{-1} \cdot \frac{1}{2} \text{aa}_0^2 + \mu a_{-1} = 0,
\]
\[
\left( \frac{G'}{G} \right)^{-1} \cdot \text{ca}_{-1} + \lambda a_{-1} + \text{aa}_0 a_{-1} = 0,
\]
\[
\left( \frac{G'}{G} \right)^0 \cdot \text{aa}_1 a_{-1} + a_{-1} + \text{ca}_0 - \mu a_1 + \frac{1}{2} \text{aa}_1^2 = 0,
\]
\[
\left( \frac{G'}{G} \right)^1 \cdot \text{aa}_c a_1 - \lambda a_1 + \text{ca}_1 = 0,
\]
\[
\left( \frac{G'}{G} \right)^2 \cdot \text{aa}_c a_1 = 0.
\]

Solving this system by Maple, we have the following two sets solutions

\[
S_1 = \left\{ a_{-1} = 0, a_0 = \frac{\lambda + \sqrt{\lambda^2 - 4\mu}}{a}, a_1 = \frac{1}{a}, c = \pm \sqrt{\lambda^2 - 4\mu}, \lambda = 0 \right\}
\]
\[
S_2 = \left\{ a_{-1} = \frac{2\mu}{a}, a_0 = \frac{\pm 4\sqrt{\mu}}{a}, a_1 = \frac{2}{a}, c = \pm \sqrt{\frac{2}{4\mu}}, \lambda = 0 \right\}
\]
Substituting the solution set $S_1$ along with Eq. (2.9) into Eq. (3.4), we have the solutions of Eq. (3.1) as follows:

When $\Omega = \lambda^2 - 4 \mu > 0$, we obtain the hyperbolic function travelling wave solutions:

$$u_1(x, t) = \frac{\lambda + \sqrt{\lambda^2 - 4 \mu}}{a} \left( \sqrt{\lambda} \cosh \left( \frac{\sqrt{\lambda}}{2} (x - ct) \right) + \frac{c_2}{\sqrt{2 \lambda}} \sinh \left( \frac{\sqrt{\lambda}}{2} (x - ct) \right) - \frac{\lambda}{2} \right).$$ (3.6)

In particular, by setting $c_2 = 0, \lambda = 0$ and $c_1 \neq 0$ in Eq. (3.6), we get

$$u_{1,1}(x, t) = \frac{2 \sqrt{\mu}}{a} \cosh \left( \sqrt{\mu} (x - ct) \right).$$ (3.7)

In particular, by setting $c_2 = 0, \lambda = 0$ and $c_1 \neq 0$ in Eq. (3.6), we get

$$u_{1,2}(x, t) = \frac{2 \sqrt{\mu}}{a} \tanh \left( \sqrt{\mu} (x - ct) \right).$$ (3.8)

When $\Omega = \lambda^2 - 4 \mu < 0$, we obtain the trigonometric function travelling wave solutions:

$$u_2(x, t) = \frac{\lambda + \sqrt{\lambda^2 - 4 \mu}}{a} \left( \sqrt{\lambda} \cos \left( \frac{\sqrt{\lambda}}{2} (x - ct) \right) + \frac{c_2}{\sqrt{2 \lambda}} \sin \left( \frac{\sqrt{\lambda}}{2} (x - ct) \right) - \frac{\lambda}{2} \right).$$ (3.9)

In particular, by setting $c_2 = 0, \lambda = 0$ and $c_1 \neq 0$ in Eq. (3.9), we get

$$u_{2,1}(x, t) = \frac{2 \sqrt{\mu}}{a} \tan \left( \sqrt{\mu} (x - ct) \right).$$ (3.10)

In particular, by setting $c_2 = 0, \lambda = 0$ and $c_1 \neq 0$ in Eq. (3.9), we get

$$u_{2,2}(x, t) = \frac{2 \sqrt{\mu}}{a} \cot \left( \sqrt{\mu} (x - ct) \right) \tag{3.11}$$

Similarly, for $S_2$:

When $\lambda^2 - 4 \mu > 0$, we obtain the hyperbolic function travelling wave solutions:

$$u_2(x, t) = \frac{\lambda - \sqrt{\lambda^2 - 4 \mu}}{a} \left( \sqrt{\lambda} \cosh \left( \frac{\sqrt{\lambda}}{2} (x - ct) \right) + \frac{c_2}{\sqrt{2 \lambda}} \sinh \left( \frac{\sqrt{\lambda}}{2} (x - ct) \right) - \frac{\lambda}{2} \right).$$ (3.12)

In particular, by setting $c_2 = 0, \lambda = 0$ and $c_1 \neq 0$ in Eq. (3.12), we get

$$u_{2,1}(x, t) = \frac{2 \sqrt{\mu}}{a} \tanh \left( \sqrt{\mu} (x - ct) \right).$$ (3.13)

When $\lambda^2 - 4 \mu < 0$, we obtain the trigonometric functions travelling wave solutions:

$$u_2(x, t) = \frac{-\lambda + \sqrt{\lambda^2 - 4 \mu}}{a} \left( \sqrt{\lambda} \cos \left( \frac{\sqrt{\lambda}}{2} (x - ct) \right) + \frac{c_2}{\sqrt{2 \lambda}} \sin \left( \frac{\sqrt{\lambda}}{2} (x - ct) \right) - \frac{\lambda}{2} \right).$$ (3.14)

In particular, by setting $c_2 = 0, \lambda = 0$ and $c_1 \neq 0$ in Eq. (3.14), we get

$$u_{2,2}(x, t) = \frac{-2 \sqrt{\mu}}{a} \left( \sqrt{\lambda} \sin \left( \frac{\sqrt{\lambda}}{2} (x - ct) \right) + \frac{c_2}{\sqrt{2 \lambda}} \cos \left( \frac{\sqrt{\lambda}}{2} (x - ct) \right) - \frac{\lambda}{2} \right).$$ (3.15)

### 3.2-Exact Solutions of the Generalized Burgers-Kdv Equation

where $p, q$ and $r$ are real constants, while $m \in \mathbb{Q}$. This equation incorporates the KdV equation ($m = 1, q = 0$), Modified KdV equation ($m = 2, q = 0$), generalized KdV equation ($q = 0$), Burgers equation ($m = 1, r = 0$), modified Burgers equation ($m = 2, r = 0$), generalized Burgers equation ($r = 0$) and the modified Burgers-KdV equation ($m = 2$), which are integrable. These equations are widely used in such fields as solid-states physics, plasma physics, fluid physics and quantum field theory. In [56], the authors solved Eq. (3.16) by extended tanh method and established some exact solutions for it. Now we will apply the extended $(e/t)\text{-Expansion Method}$ to solve Eq. (3.16). To begin with, suppose that

$$u(x, t) = u(\xi), \xi = x - ct, \tag{3.17}$$

where $\xi$ is an arbitrary constant to be determined later, the equations above converted into the following ODE

$$-cu_\xi + p u^{m+1} u_\xi + qu_\xi - ru_{\xi\xi} = 0. \tag{3.18}$$

By Integrating Eq. (3.18) and setting constant integration to zero, we get

$$-cu + \frac{p}{m+1} u^{m+1} + qu' - ru'' = 0. \tag{3.19}$$

Balancing $u'$ with $u^{m+1}$ gives $M = \frac{2}{m}$. To obtain a closed form analytic solution, the parameter $M$ should be an integer. To achieve this goal we use a transformation formula $u(\xi) = \frac{1}{\alpha} v(\xi)$. This Eq. (3.19) becomes

$$-cv^3 + \frac{p}{m+1} v^4 + \frac{2q}{m} v v' + \frac{2r}{m} v'' - \frac{2r(2m-3)}{m^2} (v')^2 = 0. \tag{3.20}$$

Balancing $vv'$ with $v^4$ gives $M = 1$. Consequently, Eq. (3.20) has the formula solution:

$$u(\xi) = a_0 + a_1 \left( \frac{\xi}{\alpha} \right) + b_{-1} \left( \frac{\xi}{\alpha} \right)^{-1}. \tag{3.21}$$

Using Eq. (3.21) along with Eq. (2.8), we derive:
Substituting Eq. (3.21), Eq. (3.22) and Eq. (3.23) into Eq. (3.20), collecting the coefficients of powers of \( \text{G} \) and setting them to zero, we obtain the following system of algebraic equations involving the parameters \( a_0, b_0 \) and \( c \) as follows:

\[
\begin{align*}
-2r^2m^2a_0^2 & \mu - 2rma_0, \mu - 2r^2a_0^2 + 2r^2a_0^2 \mu^2 + 2r^2m^2a_0^2 \\
+ 8ra_0a_1, \lambda + 16ra_0a_1, \mu + 2qma_0a_1, + 6pm^2a_0^2a_1 \\
+ 2qm^2a_1a_1 - 2cma_1a_1 - 2cm^2a_1a_1 + 2rm^2a_1a_1 \\
- 2rma_1^2 - cma_1^2 - 4r^2a_1^2 + pm^2a_1^2 - cm^2a_0 \\
- 2qm^2a_0a_0 - 2rma_0a_0 - 8rma_0a_0 \lambda^2 - 2rma_0a_1 \\
- 2qm^2a_0a_1 + 12pm^2a_0^2a_1 - 16rma_1a_1a_1 - 4ra_1^2 & = 0,
\end{align*}
\]

\[
\begin{align*}
-8ra_0a_1, \lambda + 2qm^2a_0^2 & - cma_0a_0, a_1 + 4pm^2a_0^2a_1 \\
- 16rm^2a_0a_1, \lambda + 12pm^0a_0a_0a_1 - 6rm^2a_0a_0 \\
+ 2qm^2a_0a_0a_1 - 16ra_0a_0a_1, \mu - 4rma_0a_0a_1 - 2cm^2a_0a_1 \\
- 4rma_0a_0a_1 - 2qm^2a_0a_0a_1 - 2cm^2a_0a_1 & = 0,
\end{align*}
\]

\[
\begin{align*}
8ra_0a_1, \mu & - 6rma_0a_0a_1, \mu - 4ra_0a_0a_1, \lambda + 2qm^2a_0a_0a_1 \\
+ 2qm^2a_0a_0a_1 + 6rm^2a_0a_0a_1 \\
- 4rma_0a_0a_0a_1 + 12pm^2a_0^2a_0a_1 + 4pm^2a_0^2a_1 \\
- 8rm^2a_0a_0a_1 + 2qm^2a_0a_0a_1 + 2cm^2a_0a_1^2 & = 0,
\end{align*}
\]

\[
\begin{align*}
4pm^2a_0a_1 & + 8ra_0a_1a_1 - 4rma_0a_0a_0u - 2rma_0a_0a_1 - 2cm^2a_0a_1 \\
- 2rm^2a_0a_0a_1 - 2rma_0a_0a_0u + 16raa_0a_0 + 8rma_0a_0a_0u & = 0,
\end{align*}
\]

\[
\begin{align*}
-8rm^2a_0a_0u - 4rma_0a_0a_1 & - 2cm^2a_0a_1 + 8raa_0a_0a_0u - 2cm^2a_0a_1 \\
+ 8raa_0a_0a_0u - 8rmaa_0a_0a_0u & = 0,
\end{align*}
\]

Substituting this system by Maple, we have the following sets solutions:

\[
S_1 = \left\{ a_{-1} = 0, a_0 = \frac{(qm + rlam + 4r\lambda)^2}{2pm^0(m + 4)}, \begin{array}{l}
a_1 = \frac{(q^2m^2 - r^2m^2) \lambda^2 - 8r^2m^2}{2pm^0(m + 16)}, \\
\mu = \frac{(q^2m^2 - r^2m^2) \lambda^2 - 16r^2m^2}{2pm^0(m + 16)}, \\
c = \frac{2q^2(m + 2)}{(m + 4)^2m^0} \end{array} \right\}
\]

Substituting the solution set \( S_1 \) along with Eq. (2.9) into Eq. (3.21), we have the following solutions of Eq. (3.16) as follows:

When \( \lambda^2 > 4\mu > 0 \), we obtain the hyperbolic function travelling wave solutions:

\[
u_1(x, t) = \left\{ \begin{array}{l}
\pm a_0 \pm a_1 \frac{\sqrt{c} \cosh \left( \frac{\sqrt{c}(x - ct)}{2} \right) + c \cosh \left( \frac{\sqrt{c}(x - ct)}{2} \right)}{2 \sqrt{c}} \end{array} \right\} \]

In particular, by setting \( c_1 = 0 \) and \( c_1 \neq 0 \) in Eq. (3.23), we get

\[
u_{1,1}(x, t) = \pm \frac{a_0}{m + 4m} \frac{m^2 + 1 + \tanh \left( \frac{\pm m^2}{2m} \right)}{2m} \left( \frac{x - ct}{m} \right)^{\frac{1}{m}}.
\]

While, if \( c_1 = 0 \) and \( c_2 \neq 0 \) in Eq. (3.23), we get

\[
u_{1,2}(x, t) = \pm \frac{a_0}{m + 4m} \frac{m^2 + 1 + \coth \left( \frac{\pm m^2}{2m} \right)}{2m} \left( \frac{x - ct}{m} \right)^{\frac{1}{m}}.
\]

Where \( c = \frac{2m^2 + 2m^2}{(m + 4)^2m^2} \).

When \( \lambda^2 - 4\mu < 0 \), we obtain the trigonometric functions travelling wave solutions:

\[
u_2(x, t) = \]
For $S_2$, when $\Omega = \lambda^2 - 4\mu > 0$, we obtain the hyperbolic function travelling wave solutions:

$$u_2(x, t) = \pm a_0 \pm a_1 \left( \sqrt{-\frac{c_1}{2}} \sinh \left( \frac{\sqrt{\Omega}}{2} (x - ct) \right) + c_2 \cosh \left( \frac{\sqrt{\Omega}}{2} (x - ct) \right) \right)^{-\frac{\lambda}{2}} \frac{m}{\lambda}$$ (3.27)

In particular, by setting $c_2 = 0$ and $c_1 \neq 0$ in Eq. (3.27), we get

$$u_{3, 1}(x, t) = \pm \Omega \left( 2 + \tanh \left( \frac{\frac{mq}{2r(M + 4)} (x - ct)}{m} \right) \right)^{-\frac{\lambda}{2}}$$ (3.29)

When $\Omega = \lambda^2 - 4\mu < 0$, we obtain the trigonometric functions travelling wave solutions

$$u_3(x, t) = \pm a_0 \pm a_1 \left( \sqrt{-\frac{c_1}{2}} \sin \left( \frac{\sqrt{\Omega}}{2} (x - ct) \right) + c_2 \cos \left( \frac{\sqrt{\Omega}}{2} (x - ct) \right) \right)^{-\frac{\lambda}{2}}$$ (3.30)

Where $\Omega = \frac{\lambda + 1}{\sqrt{m^4 + 3m^2 + 2}}$. | 3.3 - Exact Solutions of Lax' Fifth-Order [Lax5] Equation

This equation solved by [57] by Adomian decomposition method, [58] by extended tanh method, modified [59] by Hirota's bilinear method and established some exact solutions for it. Now we will apply the extended $(\frac{\xi}{c})$-Expansion Method to solve Eq. (3.16). To begin with, suppose that $u(x, t) = u(\xi), \xi = x - ct$.

For $S_2$, when $\Omega = \lambda^2 - 4\mu > 0$, we obtain the hyperbolic function travelling wave solutions:

$$u_2(x, t) = \pm a_0 \pm a_1 \left( \sqrt{-\frac{c_1}{2}} \sin \left( \frac{\sqrt{\Omega}}{2} (x - ct) \right) + c_2 \cos \left( \frac{\sqrt{\Omega}}{2} (x - ct) \right) \right) \frac{\lambda}{2}$$ (3.28)

In particular, by setting $c_2 = 0$ and $c_1 \neq 0$ in Eq. (3.27), we get

$$u_{3, 1}(x, t) = \pm \Omega \left( 2 + \tanh \left( \frac{\frac{mq}{2r(M + 4)} (x - ct)}{m} \right) \right)^{-\frac{\lambda}{2}}$$ (3.29)

When $\Omega = \lambda^2 - 4\mu < 0$, we obtain the trigonometric functions travelling wave solutions

$$u_3(x, t) = \pm a_0 \pm a_1 \left( \sqrt{-\frac{c_1}{2}} \sin \left( \frac{\sqrt{\Omega}}{2} (x - ct) \right) + c_2 \cos \left( \frac{\sqrt{\Omega}}{2} (x - ct) \right) \right)^{-\frac{\lambda}{2}}$$ (3.30)

Where $\Omega = \frac{\lambda + 1}{\sqrt{m^4 + 3m^2 + 2}}$.
Where \( Y = \left( \frac{G'}{G} \right) \).

Substituting the function \( \mu \) and its derivatives into the Eq. (3.33), setting the coefficients of \( \left( \frac{G'}{G} \right), (1 = 0, 1, 2, 3, 4) \) to zero, we obtain the following system of algebraic equations:

\[
\begin{align*}
\left( \frac{G'}{G} \right)^{-2} : &
240 a_2 \mu + 120 a_3 \lambda + 240 a_4 \lambda^2 + 360 a_5 \lambda^3 + 540 a_6 \lambda^4 + 60 a_7 \lambda^5 = 0, \\
\left( \frac{G'}{G} \right)^{-3} : &
150 a_2 a_3 \mu + 240 a_2 a_4 \lambda + 1180 a_2 a_5 \lambda^2 + 500 a_2 a_6 \lambda^3 + 60 a_2 a_7 \lambda^4 = 0, \\
\left( \frac{G'}{G} \right)^{-4} : &
240 a_2 a_3 \mu + 60 a_2 a_4 \lambda + 160 a_2 a_5 \lambda^2 + 120 a_2 a_6 \lambda^3 = 0, \\
\left( \frac{G'}{G} \right)^{-5} : &
120 a_2 a_3 \lambda + 120 a_2 a_4 \lambda^2 + 1480 a_2 a_5 \lambda^3 + 240 a_2 a_6 \lambda^4 + 890 a_2 a_7 \lambda^5 = 0, \\
\left( \frac{G'}{G} \right)^{-6} : &
120 a_2 a_3 \lambda + 2 a_2 a_4 \lambda^2 + 2 a_2 a_5 \lambda^3 + 4 a_2 a_6 \lambda^4 + 2 a_2 a_7 \lambda^5 = 0.
\end{align*}
\]
Solving this system by Maple, we have the following sets solutions

\[ S_1 = \left\{ a_{-2} = 0, a_{-1} = 0, a_0 = -\frac{\lambda^2}{2} - 4\mu, \right\} \]
\[ a_1 = -6\lambda, a_2 = -6, \]
\[ c = 56\mu^2 + \frac{7\lambda^4}{2} - 28\lambda^2\mu, \mu = \mu, \lambda = \lambda \]
\[ a_0 = -\frac{1}{6} \lambda^2 - 3\mu \]
\[ \pm \sqrt{40\lambda^2\mu - 80\mu^2 - 5\lambda^4 + 30c} \]
\[ a_{1} = -2\lambda, a_{2} = -2, c = c, \mu = \mu, \lambda = \lambda \]
\[ S_3 = \left\{ a_{-2} = -6\lambda^2, a_{-1} = -6\lambda \mu, a_0 = -\frac{\lambda^2}{2} - 4\mu, \right\} \]
\[ a_1 = 0, a_2 = 0, c = 56\mu^2 + \frac{7\lambda^4}{2} - 28\lambda^2\mu, \mu = \mu, \lambda = \lambda \]
\[ S_4 = \left\{ a_{-2} = -2\mu^2, a_{-1} = -2\lambda \mu, \right\} \]
\[ a_0 = -\frac{1}{6} \lambda^2 - 3\mu \pm \frac{1}{3} \sqrt{\Theta} \]
\[ a_1 = 0, a_2 = 0, c = c, \mu = \mu, \lambda = \lambda \]

Where
\[ \Theta = -5\lambda^4 + 400\lambda^2 + 1600 - 1680\mu^2 + 30c \]
Substituting the solution set $S_1$ along with Eq. (2.9) into Eq. (3.34), we have the solutions following of Eq. (3.31)

When $\lambda^2 - 4\mu > 0$, we obtain the hyperbolic function travelling wave solutions:

$$u_1(x,t) = -\frac{-\lambda^2 - 4\mu}{2}.\left(\sqrt{\frac{\lambda}{2}} \left(\frac{c_1}{(x-ct)} + c_2 \cosh \left(\frac{\sqrt{\lambda}}{2} (x-ct)\right)\right) \right)$$

$$-6\lambda \left(\frac{\sqrt{\frac{\lambda}{2}} \left(\frac{c_1}{(x-ct)} + c_2 \cosh \left(\frac{\sqrt{\lambda}}{2} (x-ct)\right)\right)}{2 \left(\frac{c_1}{(x-ct)} + c_2 \cosh \left(\frac{\sqrt{\lambda}}{2} (x-ct)\right)\right)}\right)^2 - \frac{\lambda}{2}. \nonumber \quad (3.35)$$

In particular, by setting $c_2 = 0$ and $c_1 \neq 0$ in Eq. (3.35), we get

$$u_{1,1}(x,t) = \frac{4(\mu-\lambda^2)}{2} \left(1 - 3\text{sech}^2 \left(\frac{\sqrt{\lambda}}{2} (x-ct)\right)\right). \quad (3.36)$$

while, if $c_1 = 0, c_2 \neq 0$ in Eq. (3.35), we get

$$u_{1,2}(x,t) = \frac{4(\mu-\lambda^2)}{2} \left(1 + 3\text{csch}^2 \left(\frac{\sqrt{\lambda}}{2} (x-ct)\right)\right). \quad (3.37)$$

When $\lambda^2 - 4\mu < 0$, we obtain the trigonometric function travelling wave solutions:

$$u_2(x,t) = -\frac{\lambda^2 - 4\mu}{2}.\left(\sqrt{\frac{-\lambda}{2}} \left(-\frac{c_1}{(x-ct)} + c_2 \cos \left(\frac{\sqrt{-\lambda}}{2} (x-ct)\right)\right) \right)$$

$$-6\lambda \left(\frac{\sqrt{\frac{-\lambda}{2}} \left(-\frac{c_1}{(x-ct)} + c_2 \cos \left(\frac{\sqrt{-\lambda}}{2} (x-ct)\right)\right)}{2 \left(-\frac{c_1}{(x-ct)} + c_2 \cos \left(\frac{\sqrt{-\lambda}}{2} (x-ct)\right)\right)}\right)^2 - \frac{\lambda}{2}. \nonumber \quad (3.38)$$

In particular, by setting $c_2 = 0$ and $c_1 \neq 0$ in Eq. (3.38), we get

$$u_{2,1}(x,t) = \frac{4(\mu+\lambda^2)}{2} \left(1 - 3\text{sec}^2 \left(\frac{\sqrt{-\lambda}}{2} (x-ct)\right)\right). \quad (3.39)$$

while, if $c_1 = 0, c_2 \neq 0$ in Eq. (3.39), we get

$$u_{2,2}(x,t) = \frac{4(\mu+\lambda^2)}{2} \left(1 - 3\text{csc}^2 \left(\frac{\sqrt{-\lambda}}{2} (x-ct)\right)\right). \quad (3.40)$$

When $\lambda^2 - 4\mu = 0$, we have

$$u_3(x,t) = -\frac{\lambda^2 - 4\mu}{2} - 6\lambda \left(\frac{c_1}{(x-ct)} \left(1 + \frac{\lambda}{2}\right)\right) - \frac{\lambda}{2}. \quad (3.41)$$

In particular, by setting $c_1 = 0$ and $c_2 \neq 0$ in Eq. (3.41), we get

$$u_{3,1}(x,t) = -\frac{\lambda^2 - 4\mu}{2} - 6\lambda \left(\frac{1}{(x-ct)} \left(1 + \frac{\lambda}{2}\right)\right) - 6\lambda \left(\frac{c_2}{(x-ct)} \left(1 + \frac{\lambda}{2}\right)\right)^2 \quad (3.42)$$

For $S_2$:

where $\lambda^2 - 4\mu > 0$, we obtain the hyperbolic function travelling wave solutions:

$$u_4(x,t) = a_0 \left(\sqrt{\frac{\lambda}{2}} \left(\frac{c_1}{(x-ct)} + c_2 \cosh \left(\frac{\sqrt{\lambda}}{2} (x-ct)\right)\right) \right)$$

$$-2\lambda \left(\frac{\sqrt{\frac{\lambda}{2}} \left(\frac{c_1}{(x-ct)} + c_2 \cosh \left(\frac{\sqrt{\lambda}}{2} (x-ct)\right)\right)}{2 \left(\frac{c_1}{(x-ct)} + c_2 \cosh \left(\frac{\sqrt{\lambda}}{2} (x-ct)\right)\right)}\right)^2 - \frac{\lambda}{2}. \nonumber \quad (3.43)$$

In particular, by setting $c_2 = 0$ and $c_1 \neq 0$ in Eq. (3.43), we get

$$u_{4,1}(x,t) = \pm \frac{\sqrt{80(\mu^2 - 4\lambda^2) - 5\lambda^4 + 30\lambda^2}}{2} \left(1 - 3\text{sech}^2 \left(\frac{\sqrt{\lambda}}{2} (x-ct)\right)\right). \quad (3.44)$$

while, if $c_1 = 0, c_2 \neq 0$ in Eq. (3.43), we get

$$u_{4,2}(x,t) = \pm \frac{\sqrt{80(\mu^2 - 4\lambda^2) - 5\lambda^4 + 30\lambda^2}}{2} \left(1 + 3\text{csch}^2 \left(\frac{\sqrt{\lambda}}{2} (x-ct)\right)\right). \quad (3.45)$$

When $\lambda^2 - 4\mu < 0$, we obtain the trigonometric function travelling wave solutions:

$$u_5(x,t) = a_0 \left(\sqrt{\frac{\lambda}{2}} \left(-\frac{c_1}{(x-ct)} + c_2 \cos \left(\frac{\sqrt{-\lambda}}{2} (x-ct)\right)\right) \right)$$

$$-2\lambda \left(\frac{\sqrt{\frac{\lambda}{2}} \left(-\frac{c_1}{(x-ct)} + c_2 \cos \left(\frac{\sqrt{-\lambda}}{2} (x-ct)\right)\right)}{2 \left(-\frac{c_1}{(x-ct)} + c_2 \cos \left(\frac{\sqrt{-\lambda}}{2} (x-ct)\right)\right)}\right)^2 - \frac{\lambda}{2}. \nonumber \quad (3.46)$$

In particular, by setting $c_2 = 0$ and $c_1 \neq 0$ in Eq. (3.46), we get

$$u_{5,1}(x,t) = \pm \frac{\sqrt{80(\mu^2 - 4\lambda^2) - 5\lambda^4 + 30\lambda^2}}{2} \left(1 - 3\text{sec}^2 \left(\frac{\sqrt{-\lambda}}{2} (x-ct)\right)\right). \quad (3.47)$$

while, if $c_1 = 0, c_2 \neq 0$ in Eq. (3.46), we get

$$u_{5,2}(x,t) = \pm \frac{\sqrt{80(\mu^2 - 4\lambda^2) - 5\lambda^4 + 30\lambda^2}}{2} \left(1 - 3\text{csc}^2 \left(\frac{\sqrt{-\lambda}}{2} (x-ct)\right)\right). \quad (3.48)$$

When $\lambda^2 - 4\mu = 0$, we have the solutions:

$$u_6(x,t) = a_0 - 2\lambda \left(\frac{c_2}{(x-ct)} \left(1 + \frac{\lambda}{2}\right)\right) - 2 \left(\frac{c_2}{(x-ct)} \left(1 + \frac{\lambda}{2}\right)\right)^2 \quad (3.49)$$
In particular, by setting $c_1 = 0$ and $c_2 \neq 0$ in Eq. (3.49), we get

\[ u_{6,1}(x,t) = \frac{1}{30 \tilde{a}_0^2} (10a^2(x - ct)^2 - 40 \mu(x - ct)^2 \cdot \left( -6 \lambda \mu \left( \frac{\sqrt{-\Omega}}{2} \tan \left( \frac{\sqrt{-\Omega}}{2} (x - ct) \right) - \frac{1}{2} \mu \right)^{-1} - \left( \frac{\sqrt{\lambda^2 - 4 \mu}}{2} \tan \left( \frac{\sqrt{\lambda^2 - 4 \mu}}{2} (x - ct) \right) - \frac{1}{2} \lambda \right)^{-1} \right) \]

where $a_0 = -\frac{4 \mu}{3} \frac{\lambda^2}{6} \frac{\sqrt{-80 \mu^4 + 40 \lambda^2 \mu - 8 \lambda^4 + 30 \lambda^2}}{30}$ (3.50)

For $S_2$:

When $\lambda^2 - 4 \mu > 0$, we obtain the hyperbolic function travelling wave solutions:

\[ u_{7}(x,t) = \frac{-\lambda^2 - 4 \mu}{2} \tan \left( \frac{\sqrt{-\Omega}}{2} (x - ct) \right) - \frac{1}{2} \lambda \]

where, if $c_2 = 0, c_2 \neq 0$ in Eq. (3.51), we get

\[ u_{7,1}(x,t) = \frac{-\lambda^2 - 4 \mu}{2} \tan \left( \frac{\sqrt{-\Omega}}{2} (x - ct) \right) - \frac{1}{2} \lambda \]

while, if $c_2 = 0, c_2 \neq 0$ in Eq. (3.51), we get

\[ u_{7,2}(x,t) = \frac{-\lambda^2 - 4 \mu}{2} \tan \left( \frac{\sqrt{-\Omega}}{2} (x - ct) \right) - \frac{1}{2} \lambda \]

When $\lambda^2 - 4 \mu < 0$, we obtain the trigonometric function travelling wave solutions:

\[ u_{8}(x,t) = \frac{-\lambda^2 - 4 \mu}{2} \sin \left( \frac{\sqrt{-\Omega}}{2} (x - ct) \right) - \frac{1}{2} \lambda \]

In particular, by setting $c_2 = 0$ and $c_1 \neq 0$ in Eq. (3.54), we get

\[ u_{8,1}(x,t) = \frac{-\lambda^2 - 4 \mu}{2} \sin \left( \frac{\sqrt{-\Omega}}{2} (x - ct) \right) - \frac{1}{2} \lambda \]

In particular, by setting $c_1 = 0$ and $c_2 \neq 0$ in Eq. (3.54), we get

\[ u_{8,2}(x,t) = \frac{-\lambda^2 - 4 \mu}{2} \sin \left( \frac{\sqrt{-\Omega}}{2} (x - ct) \right) - \frac{1}{2} \lambda \]

When $\lambda^2 - 4 \mu = 0$, we have the solutions:

\[ u_{9}(x,t) = \frac{-\lambda^2 - 4 \mu}{2} \sin \left( \frac{\sqrt{-\Omega}}{2} (x - ct) \right) - \frac{1}{2} \lambda \]

In particular, by setting $c_1 = 0$ and $c_2 \neq 0$ in Eq. (3.57), we get

\[ u_{9,1}(x,t) = \frac{-\lambda^2 - 4 \mu}{2} \sin \left( \frac{\sqrt{-\Omega}}{2} (x - ct) \right) - \frac{1}{2} \lambda \]

For $S_4$:

When $\lambda^2 - 4 \mu > 0$, we obtain the hyperbolic function travelling wave solutions:

\[ u_{10}(x,t) = a_0 \]

\[ u_{10,1}(x,t) = \frac{-\lambda^2 - 4 \mu}{2} \tan \left( \frac{\sqrt{-\Omega}}{2} (x - ct) \right) - \frac{1}{2} \lambda \]

\[ u_{10,2}(x,t) = \frac{-\lambda^2 - 4 \mu}{2} \tan \left( \frac{\sqrt{-\Omega}}{2} (x - ct) \right) - \frac{1}{2} \lambda \]

while, if $c_1 = 0, c_2 \neq 0$ in Eq. (3.59), we get

\[ u_{10,2}(x,t) = \frac{-\lambda^2 - 4 \mu}{2} \tan \left( \frac{\sqrt{-\Omega}}{2} (x - ct) \right) - \frac{1}{2} \lambda \]

\[ u_{10,2}(x,t) = \frac{-\lambda^2 - 4 \mu}{2} \tan \left( \frac{\sqrt{-\Omega}}{2} (x - ct) \right) - \frac{1}{2} \lambda \]
When $\lambda^2 - 4\mu < 0$, we obtain the trigonometric function travelling wave solutions:

$$u_{11}(x,t) = a_0$$

$$-6\lambda\mu\left(\frac{\sqrt{\Omega}}{2} \coth \left(\frac{\sqrt{\Omega}}{2} (x - ct)\right) - \frac{1}{2\lambda}\right)^{-1}$$

$$-\left(\frac{\sqrt{\Omega}}{2} \coth \left(\frac{\sqrt{\Omega}}{2} (x - ct)\right) - 12\lambda - 2\right). \quad (3.61)$$

In particular, by setting $c_2 = 0$ and $c_1 \neq 0$ in Eq. (3.62), we get

$$u_{11,1}(x,t) = -\frac{\lambda^2}{2} - 4\mu$$

$$-6\lambda\mu\left(\frac{\sqrt{\Omega}}{2} \tan \left(\frac{\sqrt{\Omega}}{2} (x - ct)\right) - \frac{1}{2\lambda}\right)^{-1}$$

$$-\left(\frac{\sqrt{\Omega}}{2} \tan \left(\frac{\sqrt{\Omega}}{2} (x - ct)\right) - 12\lambda - 2\right). \quad (3.63)$$

while, if $c_1 = 0, c_2 \neq 0$ in Eq. (3.62), we get

$$u_{11,2}(x,t) = -\frac{\lambda^2}{2} - 4\mu$$

$$-6\lambda\mu\left(\frac{\sqrt{\Omega}}{2} \cot \left(\frac{\sqrt{\Omega}}{2} (x - ct)\right) - \frac{1}{2\lambda}\right)^{-1}$$

$$-\left(\frac{\sqrt{\Omega}}{2} \cot \left(\frac{\sqrt{\Omega}}{2} (x - ct)\right) - 12\lambda - 2\right). \quad (3.64)$$

When $\lambda^2 - 4\mu = 0$, we have

$$u_{12}(x,t) = a_0 - 2\lambda\mu\left(\frac{c_2}{c_1 + c_2(x - ct)} - \frac{1}{2\lambda}\right)^{-1}$$

$$-2\mu^2\left(\frac{1}{(x - ct)} + \frac{1}{2\lambda}\right)^{-2}. \quad (3.65)$$

In particular, by setting $c_1 = 0$ and $c_2 \neq 0$ in Eq. (2.65), we get

$$u_{12,1}(x,t) = a_0 - 2\lambda\mu\left(\frac{1}{(x - ct)} + \frac{1}{2\lambda}\right)^{-1}$$

$$-2\mu^2\left(\frac{1}{(x - ct)} - \frac{1}{2\lambda}\right) - \frac{2\mu^2}{(x - ct)}\left(\frac{1}{2\lambda} - \frac{1}{2\lambda}\right)^{-2}. \quad (3.66)$$

5. Conclusions

In this paper, we successfully use the extended expansion method to solve some non-linear partial differential equations. This method is reliable and efficient. By comparing the results of subsection (3.1) with the results of [55], we conclude that the results: (3.7), (3.8) and (3.13) are in agreement with the results: (71), (72) and (73) of [55], respectively, when $\mu = -\frac{c^2}{4}$, this shows that our results are more general. Comparing the results of subsection (3.2) with the results of [56], we conclude that the results: (3.24), (3.25) and (3.28) are agreement with the results of [56]. This shows that our results are more general. The solutions obtained in subsection (3.3) have not been reported in the literature so far. According the results of sub-sec. (3.1) and sub-sec. (3.2), we conclude that the expansion method is more effective and general than of extended tanh method.

References


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