e-Semimaximal Submodules

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Abstract
In this paper, we introduce and study the concept e-semimaximal submodule. Also many relationships of this concept with other related concepts are given.

Key Words: e-semimaximal submodule, semimaximal submodules, δ-semimaximal submodule, multiplication modules.

1- INTRODUCTION
Let R be a commutative ring with unity and let M be a left R-module. A proper submodule N of M is called semimaximal if \( \frac{M}{N} \) is semisimple [8]. Y-Wang in [10] introduced the concept δ-semimaximal submodule, where a proper submodule N of M is called δ-semimaximal if there exist submodules \( N_1, \ldots, N_k \) of M such that \( N = \bigcap_{i=1}^{k} N_i \) and \( \frac{M}{N_i} \) is singular, for each \( i = 1, \ldots, k \). Also δ-semimaximal submodules had been studied in [6]. Recall that a submodule W of M is called essential (denoted by W ı M) if \( W \cap K = (0) \) implies K = (0), where K is a submodule of M [7]. Note that if W = (0) then W ı M if and only if M = (0), [7]. In this paper, we call that a proper submodule N of M is e-semimaximal if \( N = \bigcap_{i=1}^{k} N_i \) for some essential maximal submodules \( N_1, \ldots, N_k \). We give the main properties for this concept and its relations with other classes of submodules.

2- MAIN RESULTS

Remarks and Examples (2.1):
(1) If N is an e-semimaximal submodule in an R-module M, then N is δ-semimaximal, but the converse is not true in general.

Proof: Since N is e-semimaximal, \( N = \bigcap_{i=1}^{n} N_i \) for some essential maximal submodules \( N_i \), \( i = 1, \ldots, n \). Hence \( \frac{M}{N_i} \) is simple for all \( i = 1, \ldots, n \).

By [5,Prop.1.21.p.32], \( N_i \leq M \) implies \( \frac{M}{N_i} \) is singular, \( \forall i = 1, \ldots, n \). Thus N is δ-semimaximal.

(2) If N ı M and W is e-semimaximal, then it is not necessarily that N is e-semimaximal. For example: in the Z-module \( \mathbb{Z}_9 \), \( (3) \) is e-semimaximal, but \( (0) < (3) \) is not e-semimaximal.

(3) It is clear that the intersection of two e-semimaximal submodules is e-semimaximal.

(4) It is clear that every e-semimaximal submodule is essential and the converse may not be true, for example: in the Z-module Z_{16}, \( N = <4> \leq Z_{16} \) but it is not e-semimaximal.

(5) A homomorphic image of e-semimaximal submodule need not be e-semimaximal, for example: N = 6Z in the Z-module Z is e-semimaximal. If \( \pi: Z \longrightarrow \mathbb{Z}/N \cong \mathbb{Z}_6 \) then \( \pi(N) = (0) \) which is not e-semimaximal in \( \mathbb{Z}_6 \).

(6) Let f: M ı M’ be an R-epimorphism and let W be an e-maximal submodule of M. Then \( f^{-1}(W) \) is an e-maximal submodule of M.

Proof: It is easy, so is omitted.

(7) Let f: M ı M’ be an R-isomorphism, let N ı M. If N is an e-semimaximal submodule of M, then f(N) is an e-semimaximal submodule of M’.

Proof:
Since N is e-semimaximal, \( N = \bigcap_{i=1}^{n} N_i \) for some essential maximal submodules \( N_1, \ldots, N_n \). By essentiality of \( N_i \) and the condition f is monomorphism. We have f(\( N_i \)) ı M’, for each \( i = 1, \ldots, n \). Also f is an epimorphism and \( N_i \) is maximal if \( f(N_i) \) is maximal in M’. Beside this, f is monomorphism implies that \( f(\bigcap_{i=1}^{n} N_i) = \bigcap_{i=1}^{n} f(N_i) \).

Thus f(N) is an e-semimaximal.

(8) If N and W are isomorphic submodules of an R-module M such that N is an e-semimaximal, then it is not necessarily that W is e-semimaximal. For example: in the Z-module Z, N = 6Z is e-semimaximal and \( N \cong W = 4Z \), but W is not e-semimaximal.
Proposition (2.2):
Let $M$ be a uniform $R$-module, $N < M$. Then $N$ is $\delta$-semimaximal if and only if $N$ is $\delta$-semimaximal.
Proof: $\Rightarrow$ It is clear by Rem. (2.1(1)).
$\Leftarrow$ Since $N$ is $\delta$-semimaximal, $N = \bigcap_{i=1}^{n} W_i$, for some submodules $W_1, \ldots, W_n$ such that $\frac{M}{W_i}$ is singular simple, for each $i = 1, \ldots, n$. But $\frac{M}{W_i}$ is simple imply $W_i$ is maximal, and since $M$ is uniform, so that $W_i \leq M$, $\forall$ $i = 1, \ldots, n$. Thus $N$ is e-semimaximal.

Proposition (2.3):
Let $M$ be a nonsingular $R$-module, $N < M$. Then $N$ is e-semimaximal if and only if $N$ is $\delta$-semimaximal.
Proof: $\Rightarrow$ It follows by Rem. (2.1(1)).
$\Leftarrow$ Since $N$ is $\delta$-semimaximal, $N = \bigcap_{i=1}^{n} W_i$, for some submodules $W_1, \ldots, W_n$ where $\frac{M}{W_i}$ is singular simple, $\forall$ $i = 1, \ldots, n$. As $\frac{M}{W_i}$ is simple for each $i = 1, \ldots, n$, so that $W_i$ is maximal, $\forall$ $i = 1, \ldots, n$. On the other hand $M$ is nonsingular and $\frac{M}{W_i}$ is singular imply $W_i \leq M$ by [5]. It follows that $N$ is e-semimaximal.

Corollary (2.4):
Let $M$ be a nonsingular module over an integral domain, let $(0) \neq N < M$. If $N$ is maximal, then $N$ is e-semimaximal.
Proof: Since $N$ is maximal, $N \neq (0)$, then by [6, Th.2.6], $N$ is $\delta$-semimaximal and so by Prop. (2.3), $N$ is e-semimaximal.

Note that the condition $M$ is nonsingular is necessary condition in Cor.2.4, for example in $Z_6$ module $Z_6$, $(0) \neq N = \langle 2 \rangle < Z_6$ is maximal but not e-semimaximal.

Recall that an $R$-module $M$ is called fully prime if every proper submodule of $M$ is prime.
Recall that an $R$-module $M$ is called a multiplication module if for each submodule $N$ of $M$, there exists an ideal $I$ of $R$ such that $N = IM$.

Equivalently $M$ is a multiplication $R$-module if for each submodule $N$ of $M$, $N = [N:M]M$, [3].

Proposition (2.5):
Let $M$ be a fully prime multiplication $R$-module. Then every e-semimaximal submodule is maximal.
Proof: Let $N$ be an e-semimaximal submodule.
Then $N = \bigcap_{i=1}^{n} N_i$ for some essential maximal submodules $N_1, \ldots, N_n$. By hypothesis, $M$ is fully prime, we have $N$ is a prime submodule. As $M$ is multiplication, we have $N \supseteq N_i$ for some $t = 1, \ldots, n$. It follows that $N = N_i$ (since $N_i$ is maximal). Thus $N$ is maximal.

Proposition (2.6):
Every e-semimaximal submodule $N$ of an $R$-module $M$ is semimaximal, but not conversely.

Proof: As $N$ is e-semimaximal, $N = \bigcap_{i=1}^{n} W_i$ for some essential maximal submodules $W_1, \ldots, W_n$. Thus $\frac{M}{N}$ is isomorphic to a submodule of $\frac{M}{W_1} \oplus \cdots \oplus \frac{M}{W_n}$. But $\frac{M}{W_i} \oplus \cdots \oplus \frac{M}{W_n}$ is semisimple, hence $\frac{M}{N}$ is semisimple. Thus $N$ is semimaximal.

Example: $(\bar{0})$ in the $Z$-module $Z_6$ is semimaximal but it is not e-semimaximal.

Recall that an $R$-module is $F$-regular if every submodule $N$ of $M$ is pure; that is $IM \cap N = IN$ for each ideal $I$ of $R$, [4].
An $R$-module $M$ is called fully stable if every submodule $N$ of $M$ is stable; where $N$ is stable means that for each $R$-homomorphism $f : N \longrightarrow M$, $f(N) \subseteq N$, [1].

Proposition (2.7):
Let $M$ be a cyclic $R$-module such that $ann_R M$ is e-semimaximal. Then $M$ is fully stable.
Proof: As $ann_R M$ is e-semimaximal, so by Prop. (2.6) $ann_R M$ is semimaximal. Hence by [8] $M$ is $F$-regular and so every proper submodule is semiprime. Then by [2, Cor.(4.11),p.66], $M$ is fully stable.

Remark (2.8): Let $N < M$ such that $[N : M]$ is e-semimaximal. Then $\frac{M}{N}$ is $F$-regular, where $[N : M] = \{ r \in R : rM \subseteq N \}$. 

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Proof: By Prop.(2.6), \([N:M]\) is semimaximal. Hence by [8,Prop.(1.3.8)], \(\frac{M}{N}\) is F-regular R-module.

Proposition (2.9): Every e-semimaximal submodule is semiprime.

Proof: Let \(N\) be an e-semimaximal submodule of an R-module \(M\). Then \(N = \bigcap_{i=1}^{n} W_i\) for some essential maximal sumodules \(W_1,\ldots,W_n\) of \(M\). But every maximal submodule is prime, so that \(W_1,\ldots,W_n\) are prime submodules. Thus \(N\) is semiprime [2,Prop.(3.1),p.53].

The converse of Prop.(2.9) need not be true for example: (0) in the Z-module is semiprime but not e-semimaximal.

Remark (2.10): Let \(M\) be a cyclic R-module. If every proper submodule is e-semimaximal. Then \(M\) is fully stable.

Proof: By Prop. (2.9), every proper submodule of \(M\) is semiprime. Hence \(M\) is fully stable by [2,Prop.(4.10),p.66].

Lemma (2.11): Let \(R\) be a principal ideal domain (PID), let \(I \subset R, I \neq (0)\). Then \(I\) is a semiprime ideal if and only if \(I\) is the intersection of finite number of prime ideals.

Proposition (2.12):

Let \(R\) be a PID., let \(I \subset R, I \neq (0)\). Then \(I\) is a semiprime ideal if and only if \(I\) is an e-semimaximal ideal.

Proof: It follows directly by Lemma (2.11) and the fact that every nonzero proper prime ideal of a PID is maximal and every nonzero ideal of \(R\) is essential in \(R\).

Note that the condition \(R\) is a PID is a necessary condition in Prop. (2.12), for example in the ring \(\mathbb{Z}_2, I = \langle 6 \rangle \) is a semiprime ideal but not e-semimaximal.

Theorem (2.13):

Let \(M\) be a faithful finitely generated multiplication R-module and let \(N < M\). Then the following statements are equivalent:

(1) \(N\) is an e-semimaximal submodule of \(M\).

(2) \([N:M]\) is an e-semimaximal ideal of \(R\).

(3) \(N = IM\) for some e-semimaximal ideal \(I\) of \(R\).

Proof: (1) \(\Rightarrow\) (2) By (1), \(N = \bigcap_{i=1}^{n} W_i\) for some essential maximal submodules \(W_1,\ldots,W_n\). Then \([N:M] = \bigcap_{i=1}^{n} W_i : M\) and as \(W_i\) is maximal in \(M, \forall i = 1,\ldots,n\), we have \([W_i:M]\) is a maximal ideal in \(R, \forall i = 1,\ldots,n\). Also, since \(M\) is a faithful multiplication R-module and \(N_i \leq M\).

Hence \(\exists J_i \subset R\) such that \(N_i = J_i M, \forall i = 1,\ldots,n\), by [3,Th.(2.13)], so that \(J_i = [N_i : M] \leq R\).

(2) \(\Rightarrow\) (3) It is clear since \(N = \bigcap_{i=1}^{n} (J_i M)\). But \(M\) is faithful multiplication, so by [3,Th.(1.6)], \(N = \bigcap_{i=1}^{n} (J_i M)\). Also by [3,Th.(3.1)], [3,Th.(2.13)] \(J_i M\) is maximal in \(M\) and \(J_i M \leq M, \forall i = 1,\ldots,n\). Thus \(N\) is an e-semimaximal submodule of \(M\).

Corollary (2.14):

Let \(M\) be a finitely generated faithful multiplication module over a PID \(R\), let \((0) \neq N < M\). Then \(N\) is e-semimaximal if and only if \(N\) is semiprime.

Proof: \(\Rightarrow\) It is clear by Prop.(2.9).

\(\Leftarrow\) Since \(N\) is a semiprime submodule, then \([N:M]\) is a semiprime ideal of \(R\). But \(M\) is a R multiplication R-module and \(N \neq (0)\), so that \([N:M] \neq (0)\). Hence by Prop. (2.12), \([N:M]\) is an e-semimaximal ideal of \(R\). Thus \(N\) is an e-semimaximal submodule of \(M\) by Th. (2.13).

Recall that the Jacobson radical of an R-module \(M\) (denoted by \(J(M)\) or \(\text{Rad} M\)) is the intersection of all maximal submodules of \(M\), if \(M\) has maximal submodules and \(J(M) = M\) if \(M\) has no maximal submodules [7]. Equivalently, \(J(M) = \sum_{U \in M} U\), where \(U\) is a small submodules of \(M\), [7]. Also \(U\) is a small submodules of \(M\) (denoted by \(U \ll M\)) if \(U\) is a proper submodule of \(M\) and \(U + W \neq M\) for any proper submodule \(W\) of \(M\), [7]. D.X.Zhou and X.R.Zhang introduced \(\text{Rad}_e M\), where \(\text{Rad}_e M = \bigcap \{N < M: N\) is maximal e

Similarly we define the concept e-J(M) (or e-Rad M) as follows: if M has e-semimaximal submodule then e-Rad M = ∩ {N: N is an e-semimaximal submodule of M} and e-Rad M = M if M has no e-semimaximal submodule.

However the following proposition shows that Rad,M and e-Rad M are identical.

Proposition (2.15):
For an R-module M, Rad,M = e-Rad M.

Proof: Let m ∈ Rad,M. Then m belongs to any essential maximal submodule of M, so m belongs to any finite intersection of essential maximal submodules. Hence m belongs to any semimaximal submodule and so m ∈ e-Rad M; that is Rad,M ⊆ e-Rad M.

Now let m ∈ e-Rad M; hence m is in any semimaximal submodule of M. But every essential maximal submodule of M is semimaximal, so that m ∈ ∩ {N < M: N is maximal in M} = Rad,M.

Hence e-Rad M ⊆ Rad,M. Thus e-Rad M = Rad,M.

Theorem (2.16):
Let M be a faithful finitely generated multiplication R-module. Then Rad,M= (Rad,R)M.

Proof: Since Rad,M = e-Rad,M, so that e-Rad,M = ∩ {N: N is semimaximal submodule in M}. But M is a faithful finitely generated multiplication R-module, so every semimaximal submodule N of M, N = IM for some semimaximal ideal I of R by Th. (2.13). Thus e-Rad M = ∩ {IM:I is semimaximal ideal of R}. But M is faithful multiplication, so that ∩(IM) = (∩I)M by [3,Th.(1.6)]. Hence e-Rad M = (e-Rad,R)M = (Rad,R)M.

Theorem (2.17):
Let M be an R-module. Consider the following statements:
(1) M is Artinian.
(2) M satisfies descending chain condition on e-small submodules and on e-semimaximal submodules.
(3) M satisfies descending chain condition on e-small submodules and Rad,M is e-semimaximal submodule.

Then (1) ⇒ (2) ⇒ (3) and (3) ⇒ (1) if Rad,M is Artinian.

Proof: (1) ⇒ (2) It is clear.

(2) ⇒ (3) Since M satisfies descending chain condition on e-semimaximal, M has a minimal e-semimaximal submodule say N. So N = ∩ N_i, N_i is essential maximal for each i = 1, …, n. Hence Rad,M ≤ N. So if Rad,M = M, then M = N which is a contradiction. Thus M ≠ Rad,M. If P is any maximal essential submodule of M, then N ∩ P is e-semimaximal submodule. But N is minimal e-semimaximal, so N = N ∩ P, thus N ≤ P. Hence N ≤ Rad,M. But Rad,M ≤ N. So that N = Rad,M. Thus Rad,M is e-semimaximal submodule.

(3) ⇒ (1) Rad,e M is Artinian. As Rad,e M is e-semimaximal, M ≠ Rad,e M and Rad,e M = ∩ P_i for some P_i essential maximal submodule of M for each i = 1,…, n. But P_i ≤ M, implies M P_i is singular, [4,Proposition 1.21,p.32]. Also P_i is maximal, implies M P_i is simple. Since M Rad,e M = ∩ P_i submodule of M P_i ⊕ M P_i, and M P_i ⊕ M P_i is a semisimple module so M P_i ⊕ M P_i is Artinian.

Hence M Rad,e M is Artinian. But Rad,M is Artinian, it follows that M is Artinian.

Next we have that:

Proposition (2.18):
Let I be an e-semimaximal ideal of a ring R. Then I[X] is an e-semimaximal ideal of R[X], provide that every ideal of R[X] has the form K[X] for some ideal K of R.

Proof: Since I is e-semimaximal, I = ∩ J_i for some essential maximal ideals J_i, …, J_n of R and then I[X] = ∩ (J_i[X]). It follows that I[X] = ∩ (J_i[X]). But J_i ⊆ R, ∀ i = 1, …, n implies that J_i[X] ≤ R[X], e by [9,Exc.30,p.116].

On the other hand for any i = 1, …, n, J_i[X] is a maximal ideal in R, since if there exists an ideal W_i[X] in R[X] such that J_i[X] ≤ W_i[X] for some ideal W in R. Then J_i ⊆ W. Hence J_i = W and so J_i[X] = W[X]. Thus I[X] is an e-semimaximal ideal in R[X].
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