4-cordiality of Some New Path Related Graphs

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Abstract—For an Abelian Group < A, * > a graph G = (V(G), E(G)) is said to be A-cordial if there is a mapping f:V(G)→A which satisfies the conditions |v(a)|-|v(b)|≤1 and |e(a)|-|e(b)|≤1, for all a,b ∈ A, when the edge e=uv is labeled as f(u)*f(v). Where v(a) is the number of vertices with label a and e(a) is the number of edges with label a. If we consider an Abelian Group < A, * > =< Z\(_k\), +\(_k\) > then it is called k-cordial labeling. In this research paper we prove that Z-P\(_m\), braid graph B(n), triangular ladder TL\(_n\), and irregular quadrilateral snake I(QS\(_n\)) are k-cordial for all n.

Keywords—A-cordial Labeling; Z-P\(_m\); Braid Graph B(n); Triangular Ladder TL\(_n\); Irregular Quadrilateral Snake I(QS\(_n\)).

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I. INTRODUCTION

In this research paper we consider finite, connected, undirected and simple graphs. In the graph G=({V(G), E(G)}) the cardinality of the vertex set is called order of G and the cardinality of the edge set is called the size of G. They are denoted by |V(G)| and |E(G)| respectively. In Graph Labeling we assign numerical values to vertices or edges or both subject to certain conditions.

Definition 1.1 Let < A, * > be any Abelian group. A graph is said to be A-cordial if there is a mapping f:V(G)→A which satisfies the following two conditions when the edge e=uv is labeled as f(u)*f(v)

(i) |v(a)|-|v(b)|≤1; for all a,b ∈ A,
(ii) |e(a)|-|e(b)|≤1; for all a,b ∈ A.

Where,

v(a)=the number of vertices with label a;

v(b)=the number of vertices with label b;

e\(_l\)(a)=the number of edges with label a;

e\(_l\)(b)=the number of edges with label b.

We note that if A=< Z\(_4\), +\(_4\) > that is additive group of modulo 4 then the labeling is known as 4-cordial labeling.

Here, we consider < Z\(_4\), +\(_4\) > that is additive group of modulo 4 then the labeling is known as 4-cordial labeling.

The concept of A-cordial labeling was introduced by Hovey [3] and proved the following results:

- All the connected graphs are 3-cordial.
- All the trees are 3-cordial and 4-cordial.
- Cycles are k-cordial for all odd k.

Youssef [10] obtained the following results:

- C\(_{2k+1}\) is not (2k + 1)-cordial for k > 1.
- K\(_n\) is 4-cordial if n ≤ 6.
- C\(_n\) is 4-cordial if n ≢ 2(mod4).
- K\(_{m,n}\) is 4-cordial if m or n ≢ 2(mod4).

Rathod and Kanani [4] proved the following results:

- All the wheels W\(_n\) are 4-cordial.
- All the fans f\(_n\) are 4-cordial.
- All the friendship graphs F\(_n\) are 4-cordial.
- All the gear graphs G\(_n\) are 4-cordial.
- All the double fans Df\(_n\) are 4-cordial.
- All the helm graphs H\(_n\) are 4-cordial.

Rathod and Kanani [5] also proved the following results:

- The middle graph M(P\(_n\)) of path P\(_n\) is 4-cordial.
- The total graph T(P\(_n\)) of path P\(_n\) is 4-cordial.
- The splitting graph S(P\(_n\)) of path P\(_n\) is 4-cordial.
- The square graph P\(_n^2\) of path P\(_n\) is 4-cordial.
- The triangular snake T\(_n\) is 4-cordial.

In [6] Rathod and Kanani have derived the following results:

- The square graph of Path P\(_n^2\) is k-cordial.
- The pan graph C\(_{n^4+1}\) is k-cordial for all even k and
  n = k + j, 0 ≤ j ≤ k - 1.
- The pan graph C\(_{n^4+1}\) is k-cordial for all even k and
\[ n = 2tk + j, \text{ where } t \in \text{NU}\{0\}\text{ and } 0 \leq j \leq k - 1. \]

- The pan graph \( C_n^{(i)} \) is k-cordial for all even \( k \) and \( n = 2tk + j, \text{ where } t \in \text{N} \text{ and } 0 \leq j \leq k - 1. \)

We consider the following useful definitions to understand the results of this research paper.

**Definition 1.2** The graph \( Z-P_n \) is obtained from the pair of paths \( P'_n \) and \( P''_n \). Let \( v_1, v_2, \ldots, v_n \) be the vertices of path \( P'_n \) and \( u_1, u_2, \ldots, u_n \) are the vertices of path \( P''_n \). To find \( Z-P_n \) join \( i^{th} \) vertex of path \( P'_n \) with \((i + 1)^{th} \) vertex of path \( P''_n \) for all \( 1 \leq i \leq n - 1 \).

**Definition 1.3** The Braid Graph \( B(n) \) is obtained from the pair of paths \( P'_n \) and \( P''_n \). Let \( v_1, v_2, \ldots, v_n \) be the vertices of path \( P'_n \) and \( u_1, u_2, \ldots, u_n \) are the vertices of path \( P''_n \). To find braid graph join \( i^{th} \) vertex of path \( P'_n \) with \((i + 1)^{th} \) vertex of path \( P''_n \) and \( i^{th} \) vertex of path \( P''_n \) with \((i + 2)^{th} \) vertex of path \( P''_n \) with the new edges for all \( 1 \leq i \leq n - 2 \).

**Definition 1.4** The Triangular Ladder \( TL_n \) is obtained from the ladder \( L_n = P_n \times P_2 \) \((n \geq 2)\) by adding the edges \( u_i v_{i+1} \) for all \( 1 \leq i \leq n-1 \), where the consecutive vertices of two copies of paths are \( v_1, v_2, \ldots, v_n \) and \( u_1, u_2, \ldots, u_n \) and the edges are \( u_i v_i \).

**Definition 1.5** The Irregular Quadrilateral Snake \( I(QS_n) \) is obtained from the path \( P_n \). Let \( u_1, u_2, \ldots, u_n \) be the vertices of path \( P_n \) and \( v_1, v_2, \ldots, v_{n-2} \) \& \( w_1, w_2, \ldots, w_{n-2} \) are the newly added vertices. To obtain irregular quadrilateral snake join the vertices \( u_i v_i, w_{i+2} u_{i+2} \) \& \( v_i w_i \) for all \( 1 \leq i \leq n - 2 \).

Here, all terminologies are considered from Gross and Yellen[2].

**II. MAIN RESULTS**

**Theorem 2.1** The graph \( Z-P_n \) is 4-cordial for all \( n \).

**Proof.** Let \( G=Z-P_n \) be the graph obtained from the pair of paths \( P'_n \) and \( P''_n \). Let \( v_1, v_2, \ldots, v_n \) be the vertices of path \( P'_n \) and \( u_1, u_2, \ldots, u_n \) are the vertices of path \( P''_n \). To find \( Z-P_n \) join \( i^{th} \) vertex of path \( P'_n \) with \((i + 1)^{th} \) vertex of path \( P''_n \) for all \( 1 \leq i \leq n - 1 \). We note that \( |V(G)| = 2n \) and \( |E(G)| = 3n - 3 \).

Define 4-cordial labeling \( f: V(G) \rightarrow \mathbb{Z}_4 \) as follows:

\[
\begin{align*}
f(v_i) &= 0; & i &= 1, 6(mod8); \\
f(v_i) &= 1; & i &= 4, 7(mod8); \\
f(v_i) &= 2; & i &= 2, 5(mod8); \\
f(v_i) &= 3; & i &= 0, 3(mod8); & 1 \leq i \leq n, \\
f(u_i) &= 0; & i &= 4, 7(mod8); \\
f(u_i) &= 1; & i &= 2, 5(mod8); \\
f(u_i) &= 2; & i &= 0, 3(mod8); \\
\end{align*}
\]

Define 4-cordial labeling \( f: V(G) \rightarrow \mathbb{Z}_4 \) we consider the following two cases:

**Case 1:** If \( n \leq 3 \).

Let \( n = 8p + q, \) \( p, q \in \text{NU}\{0\}\).

**TABLE 1**

<table>
<thead>
<tr>
<th>( q )</th>
<th>Vertex conditions</th>
<th>Edge conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( v_i = v_i(1) = 1 )</td>
<td>( e_i(0) = e_i(1) + 1 = e_i(2) + 1 = e_i(3) + 1 )</td>
</tr>
<tr>
<td>1</td>
<td>( v_i = v_i(1) + 1 )</td>
<td>( e_i(0) = e_i(1) = e_i(2) = e_i(3) )</td>
</tr>
<tr>
<td>2</td>
<td>( v_i = v_i(2) + 1 )</td>
<td>( e_i(0) = e_i(1) = )e_i(2) = e_i(3) + 1 )</td>
</tr>
<tr>
<td>3</td>
<td>( v_i = v_i(3) + 1 )</td>
<td>( e_i(0) = e_i(1) = e_i(2) = e_i(3) + 1 )</td>
</tr>
<tr>
<td>4</td>
<td>( v_i = v_i(3) + 2 )</td>
<td>( e_i(0) = e_i(1) + 1 = e_i(2) + 1 = e_i(3) + 1 )</td>
</tr>
<tr>
<td>5</td>
<td>( v_i = v_i(3) + 3 )</td>
<td>( e_i(0) = e_i(1) = e_i(2) = e_i(3) + 1 )</td>
</tr>
<tr>
<td>6</td>
<td>( v_i = v_i(3) + 4 )</td>
<td>( e_i(0) = e_i(1) = e_i(2) = e_i(3) + 1 )</td>
</tr>
<tr>
<td>7</td>
<td>( v_i = v_i(3) + 5 )</td>
<td>( e_i(0) = e_i(1) = e_i(2) = e_i(3) + 1 )</td>
</tr>
</tbody>
</table>

From the Table 1 we can see that the labeling pattern defined above satisfies all the conditions of 4-cordiality. Hence, the graph \( Z-P_n \) is 4-cordial for all \( n \).

**Illustration 2.2** The graph \( Z-P_n \) and its 4-cordial labeling is shown in Figure 1.

\[ \text{Fig. 1 4-cordial labeling of } Z-P_n \]

**Theorem 2.3** The Braid graph \( B(n) \) is 4-cordial for all \( n \).

**Proof.** Let \( G = B(n) \) be the braid graph obtained from the pair of paths \( P'_n \) and \( P''_n \). Let \( u_1, u_2, \ldots, u_n \) be the vertices of path \( P'_n \) and \( v_1, v_2, \ldots, v_{n-2} \) \& \( w_1, w_2, \ldots, w_{n-2} \) are the newly added vertices. To find braid graph join \( i^{th} \) vertex of path \( P'_n \) with \((i + 1)^{th} \) vertex of path \( P''_n \) and \( i^{th} \) vertex of path \( P''_n \) with \((i + 2)^{th} \) vertex of path \( P''_n \) with the new edges for all \( 1 \leq i \leq n - 2 \). We note that \( |V(G)| = 2n \) and \( |E(G)| = 4n - 5 \).

Define 4-cordial labeling \( f: V(G) \rightarrow \mathbb{Z}_4 \) we consider the following two cases:
Cordial labeling is shown in Fig. 2 for all $n \geq 4$. Let $n = 4p + q$, $p, q \in \mathbb{N}\{0\}$.

**Table 2**

<table>
<thead>
<tr>
<th>q</th>
<th>Vertex conditions</th>
<th>Edge conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>0,2</td>
<td>$v_f(0) = v_f(1) = v_f(2)$</td>
<td>$e_f(0) = e_f(1) = e_f(2)$</td>
</tr>
<tr>
<td>1,3</td>
<td>$v_f(0) = v_f(1) = v_f(2) + 1 = v_f(3) + 1$</td>
<td>$e_f(0) = e_f(1) = e_f(2) = e_f(3) + 1$</td>
</tr>
</tbody>
</table>

From the Table 2 we can see that the labeling pattern defined above satisfies all the conditions of 4-cordiality. Hence, the braid graph $B(n)$ is 4-cordial for all $n$.

**Illustration 2.4** (a) The Braid graph $B(3)$ and its 4-cordial labeling is shown in Figure 2.

(b) The Braid graph $B(7)$ and its 4-cordial labeling is shown in Figure 3.

**Theorem 2.5** The Triangular Ladder $TL_n$ is 4-cordial for all $n$.

**Proof.** Let $G = TL_n$ be the triangular ladder obtained from the ladder $L_n = P_2 \times P_3$ ($n \geq 2$) by adding the edges $u_i v_{i+1}$ for all $1 \leq i \leq n-1$, where the consecutive vertices of two copies of paths are $v_i, v_2, \ldots, v_n$ and $u_1, u_2, \ldots, u_n$ and the edges are $u_i v_i$. We note that $|V(G)| = 2n$ and $|E(G)| = 4n - 3$.

Define 4-cordial labeling $f : V(G) \rightarrow Z_4$ as follows:

<table>
<thead>
<tr>
<th>$f(u_i)$</th>
<th>$f(v_i)$</th>
<th>$f(v_{i+1})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i \equiv 0, 2 \pmod{4}$</td>
<td>$i \equiv 1, 3 \pmod{8}$</td>
<td>$i \equiv 1, 3 \pmod{8}$</td>
</tr>
</tbody>
</table>

From the Table 3 we can see that the labeling pattern defined above satisfies all the conditions of 4-cordiality. Hence, triangular ladder $TL_n$ is 4-cordial for all $n$.

**Illustration 2.6** The Triangular Ladder $TL_6$ and its 4-cordial labeling is shown in Figure 4.

**Theorem 2.7** The Irregular Quadrilateral Snake $IQ(S_n)$ is 4-cordial for all $n$.

**Proof.** Let $G = IQ(S_n)$ be the irregular quadrilateral snake of the path $P_n$. Let $u_1, u_2, \ldots, u_n$ be the vertices of path $P_n$ and $v_i, v_2, \ldots, v_{n-2}$ & $w_i, w_2, \ldots, w_{n-2}$ are the newly added vertices. To find irregular quadrilateral snake join the vertices $u_i v_i, w_i u_{i+1}$ and $v_i w_i$ for all $1 \leq i \leq n - 2$. We note that $|V(G)| = 3n-4$ and $|E(G)| = 4n - 7$. 

Table 3

<table>
<thead>
<tr>
<th>q</th>
<th>Vertex conditions</th>
<th>Edge conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$v_f(0) = v_f(1) = v_f(2) = v_f(3)$</td>
<td>$e_f(0) = e_f(1) = e_f(2) = e_f(3)$</td>
</tr>
<tr>
<td>1</td>
<td>$v_f(0) + 1 = v_f(1) + 1 = v_f(2) = v_f(3)$</td>
<td>$e_f(0) + 1 = e_f(1) = e_f(2) + 1 = e_f(3) + 1$</td>
</tr>
</tbody>
</table>

Fig. 2 4-cordial labeling of Braid graph $B(3)$

Fig. 4 4-cordial labeling of Triangular Ladder $TL_6$
Define 4-cordial labeling \( f : V(G) \rightarrow \mathbb{Z}_4 \) as follows:

\[
\begin{align*}
    f(u_i) &= 0; & i & \equiv 0 \pmod{4}; & 1 \leq i \leq n, \\
    f(u_i) &= 1; & i & \equiv 2 \pmod{4}; \\
    f(u_i) &= 2; & i & \equiv 1 \pmod{4}; \\
    f(u_i) &= 3; & i & \equiv 3 \pmod{4}; \\
    f(v_i) &= 0; & i & \equiv 3 \pmod{4}; \\
    f(v_i) &= 1; & i & \equiv 1 \pmod{4}; \\
    f(v_i) &= 2; & i & \equiv 0, 2 \pmod{4}; \\
    f(w_i) &= 0; & i & \equiv 1 \pmod{4}; \\
    f(w_i) &= 1; & i & \equiv 3 \pmod{4}; \\
    f(w_i) &= 2; & i & \equiv 0, 2 \pmod{4}.
\end{align*}
\]

Let \( n = 8p + q \), where \( p, q \in \mathbb{N} \cup \{0\} \).

### Table 4

<table>
<thead>
<tr>
<th>( q )</th>
<th>Vertex conditions</th>
<th>Edge conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>0,4</td>
<td>( v_f(0) = v_f(1) = v_f(2) = v_f(3) )</td>
<td>( e_f(0) + 1 = e_f(1) + 1 = e_f(2) + 1 = e_f(3) )</td>
</tr>
<tr>
<td>1,5</td>
<td>( v_f(0) = v_f(1) = v_f(2) + 1 = v_f(3) )</td>
<td>( e_f(0) + 1 = e_f(1) + 1 = e_f(2) + 1 = e_f(3) )</td>
</tr>
<tr>
<td>2,6</td>
<td>( v_f(0) + 1 = v_f(1) = v_f(2) + 1 = v_f(3) )</td>
<td>( e_f(0) = e_f(1) + 1 = e_f(2) + 1 = e_f(3) + 1 )</td>
</tr>
<tr>
<td>3,7</td>
<td>( v_f(0) + 1 = v_f(1) = v_f(2) + 1 = v_f(3) + 1 )</td>
<td>( e_f(0) = e_f(1) + 1 = e_f(2) + 1 = e_f(3) + 1 )</td>
</tr>
</tbody>
</table>

From the Table 4 we can see that the labeling pattern defined above satisfies all the conditions of 4-cordiality. Hence, irregular quadrilateral snake \( IQ(S_n) \) is 4-cordial for all \( n \).

**Illustration 2.8** The Irregular Quadrilateral Snake \( IQ(S_{11}) \) and its 4-cordial labeling is shown in Figure 5.

### III. Conclusions

Graph labeling technique is a wide area of research. In this research paper, we investigate some new results on 4-cordiality of graphs. For better understanding of labeling pattern, we have given some illustration. To investigate more graph families which admit k-cordial labeling is an open area of research.

### References