CI-algebras and its Fuzzy Ideals

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ABSTRACT

In this paper we develop the idea of fuzzy ideals in cartesian product of CI-algebras and obtain some new results. Finally we investigate how to extend a given fuzzy ideal of a CI-algebra to that of another CI-algebra.

Keywords: CI-algebra, Ideals, Fuzzy ideal

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1. INTRODUCTION

In 1966, Y. Imai and K. Iseki [2] introduced the notion of a BCK-algebra. There exist several generalizations of BCK-algebras, such as BCI-algebras [3], BCH-algebras [1], BH-algebras [4], d-algebras [8], etc. In [5], H.S. Kim and Y.H. Kim introduced the notion of a BE-algebra as a dualization of a generalization of a BCK-algebra. As a generalization of Be-algebras, B.L. Meng [7] introduced the notion of CI-algebras and discussed its important properties. The concept of fuzzification of ideals in CI-algebra have introduced by Samy M. Mostafa, Mokthar A. Abdel Naby, Osama R. Elgendy [10]. In this paper we develop the idea of fuzzy ideals in cartesian product of CI-algebras and obtain some new results. By establishing that if X is a CI-algebra then F(X), the class of all functions f : X → X, is also a CI-algebra, we extend a given fuzzy ideal of x to that of F(X).

2. PRELIMINARIES

Definition 2.1. ([7]) A system (X, *, 1) consisting of a non-empty set X, a binary operation * and a fixed element 1, is called a CI–algebra if the following conditions are satisfied:

1. (CI 1) x * x = 1
2. (CI 2) 1 * x = x
3. (CI 3) x * (y * z) = y * (x * z)

for all x, y, z ∈ X

Example 2.2. Let X = R⁺ = {x ∈ R : x > 0}

For x, y ∈ X, we define

x * y = y * x

Then (X, *, 1) is a CI–algebra

Definition 2.3. ([7]) A non-empty subset I of a CI–algebra X is called an ideal of X if

1. x ∈ X and a ∈ I ⇒ x * a ∈ I;
2. x ∈ X and a, b ∈ I ⇒ (a * (b * x)) * x ∈ I.

Lemma 2.4. ([7]) In a CI–algebra following results are true:

1. x * ((x * y) * y) = 1
Lemma 2.5. ([6]) - (i) Every ideal I of X contains 1

(ii) If I is an ideal of X then \((a \ast x) \ast x \in I\) for all \(a \in I\) and \(x \in X\)

(iii) If \(I_1\) and \(I_2\) are ideals of X then so is \(I_1 \cap I_2\).

Theorem 2.6. ([9]) - Let \((X; \ast, 1)\) be a system consisting of a non-empty set X, a binary operation \(\ast\) and a fixed element 1. Let \(Y = X \times X\). For \(\mu = (x_1, x_2)\), \(\nu = (y_1, y_2)\) a binary operation “\(\bigcirc\)” is defined in Y as

\[\mu \bigcirc \nu = (x_1 \ast y_1, x_2 \ast y_2)\]

Then \((Y; \bigcirc, (1, 1))\) is a CI-algebra iff \((X; \ast, 1)\) is a CI-algebra.

Theorem 2.7. ([9]) - Let A and B be subsets of a CI-algebra X. Then \(A \times B\) is an ideal of \(Y = X \times X\) iff A and B are ideals of X.

3. FUZZY IDEALS.

Definition (3.1) ([10]) - Let \((X; \ast, 1)\) be a CI-algebra and let \(\mu\) be a fuzzy set in X. Then \(\mu\) is called a fuzzy ideal of X if it satisfies the following conditions:

\begin{align*}
(1) \quad (\forall x, y \in X) \ (\mu(x \ast y) \geq \mu(y)), \\
(2) \quad (\forall x, y, z \in X) \ (\mu((x \ast (y \ast z)) \ast z) \geq \min \{\mu(x), \mu(y)\})
\end{align*}

Theorem (3.2) ([10]) - Let \(\mu\) be a fuzzy set in a CI-algebra \((X; \ast, 1)\). and let \(U(\mu; \alpha) = \{x \in X : \mu(x) \geq \alpha\}\) where \(\alpha \in [0, 1]\).

Then \(\mu\) is a fuzzy ideal of X iff \((\forall \alpha \in [0, 1]) \ (U(\mu; \alpha) \neq \emptyset \Rightarrow U(\mu; \alpha)\) is an ideal of X).

Proposition (3.3) ([10]) - Let \(\mu\) be a fuzzy ideal of a CI-algebra \((X; \ast, 1)\). Then

\begin{align*}
(a) \quad \mu(1) \geq \mu(x)\ \text{for all} \ x \in X; \\
(b) \quad \mu((x \ast y) \ast y) \geq \mu(x)\ \text{for all} \ x, y \in X; \\
(c) \quad x, y \in X\ \text{and} \ x \leq y \Rightarrow \mu(x) \leq \mu(y),
\end{align*}

i.e., a fuzzy ideal \(\mu\) is order preserving.

Now we establish some results for fuzzy ideals on Cartesian product of CI-algebras.

Theorem (3.4) - Let \(\mu\) be a fuzzy set on a CI-algebra X and let \(Y = X \times X\). Let \(\mu_1, \mu_2, \mu_3\) be fuzzy sets on Y defined as

\[\mu_1(x, y) = \mu(x)\]

\[\mu_2(x, y) = \mu(y)\]
\[ \mu_3(x, y) = \min \{ \mu(x), \mu(y) \} \]

Then, (a) \( \mu_1 \) is a fuzzy ideal of \( Y \) iff \( \mu \) is a fuzzy ideal of \( X \);

(b) \( \mu_2 \) is a fuzzy ideal of \( Y \) iff \( \mu \) is a fuzzy ideal of \( X \);

(c) \( \mu_3 \) is a fuzzy ideal of \( Y \) iff \( \mu \) is a fuzzy ideal of \( X \).

**Proof :-** For any real \( \alpha \in [0, 1] \), let

\[
U(\mu; \alpha) = \{ x \in X : \mu(x) \geq \alpha \};
\]

\[
U_1(\mu_1; \alpha) = \{ (x, y) \in Y : \mu_1(x, y) = \mu(x) \geq \alpha \};
\]

\[
U_2(\mu_2; \alpha) = \{ (x, y) \in Y : \mu_2(x, y) = \mu(y) \geq \alpha \};
\]

and \( U_3(\mu_3; \alpha) = \{ (x, y) \in Y : \mu_3(x, y) \geq \alpha \}; \)

Then we see that

\[
U_1(\mu_1; \alpha) = U(\mu; \alpha) \times X
\]

\[
U_2(\mu_2; \alpha) = X \times U(\mu; \alpha)
\]

\[
U_3(\mu_3; \alpha) = U(\mu; \alpha) \times U(\mu; \alpha)
\]

Now using theorem (2.7) we see that

(i) \( U_1(\mu_1; \alpha) \) is an ideal in \( Y \) iff \( U(\mu; \alpha) \) is an ideal in \( X \)

(ii) \( U_2(\mu_2; \alpha) \) is an ideal in \( Y \) iff \( U(\mu; \alpha) \) is an ideal in \( X \)

(iii) \( U_3(\mu_3; \alpha) \) is an ideal in \( Y \) iff \( U(\mu; \alpha) \) is an ideal in \( X \).

for all real \( \alpha \in [0, 1] \).

Using theorem (3.2) we get the result.

**Definition (3.5):** Let \( \mu \) be a fuzzy set in \( Y = X \times X \). Let \( \mu_1 \) and \( \mu_2 \) be fuzzy sets defined in \( X \) as

\[ \mu_1(x) = \mu(x, 1) \]

and \( \mu_2(x, y) = \mu(1, x) \)

**Theorem (3.6):** \( \mu \) is a fuzzy ideal of \( Y \) iff \( \mu_1 \) and \( \mu_2 \) are fuzzy ideals of \( X \).

**Proof :-** Let \( \mu \) be a fuzzy ideal of \( Y \). Then

\[ U(\mu; \alpha) = \{ (x, y) \in Y : \mu(x, y) \geq \alpha \} = A \times B \ (say) \]

is an ideal in \( Y \). This means that \( A \) and \( B \) are ideal in \( X \) [theorem (2.7)]. So \( 1 \in A \cap B \).

Now we prove that,

\[ U_1(\mu_1; \alpha) = \{ x \in X : \mu_1(x) \geq \alpha \} = A \]

and

\[ U_2(\mu_2; \alpha) = \{ x \in X : \mu_2(x) \geq \alpha \} = B. \]
We see that, \( x \in A, 1 \in B \Leftrightarrow (x, 1) \in U(\mu; \alpha) \Leftrightarrow \mu(x, 1) \geq \alpha \Leftrightarrow \mu_1(x) \geq \alpha \Leftrightarrow x \in U_1(\mu_1; \alpha) \)  Hence \( A = U_1(\mu_1; \alpha) \). Similarly we can prove that \( B = U_2(\mu_2; \alpha) \).

Thus we see that \( U_1(\mu_1; \alpha) \) and \( U_2(\mu_2; \alpha) \) are ideals in \( X \) for every \( \alpha \in [0, 1] \). Hence \( \mu_1 \) and \( \mu_2 \) are fuzzy ideals in \( X \).

Conversely, suppose that \( \mu_1 \) and \( \mu_2 \) are fuzzy ideals in \( X \). Then, \( U_1(\mu_1; \alpha) = \{ x \in X : \mu_1(x) \geq \alpha \} \) and \( U_2(\mu_2; \alpha) = \{ x \in X : \mu_2(x) \geq \alpha \} \) are ideals in \( X \) for every \( \alpha \in [0, 1] \).

So \( 1 \in U_1(\mu_1; \alpha) \cap U_2(\mu_2; \alpha) \) which means that \( \mu(1, 1) \geq \alpha \) \[\text{def.}(3.5)\]

Now \( U(\mu; \alpha) = \{ (x, y) \in Y : \mu(x, y) \geq \alpha \} = A \times B \) (Say) contains \( (1, 1) \).

We see that, \( x \in U_1(\mu_1; \alpha) \Leftrightarrow \mu_1(x) \geq \alpha \Leftrightarrow \mu(x, 1) \geq \alpha \Leftrightarrow (x, 1) \in U(\mu; \alpha) \Leftrightarrow x \in A, 1 \in B \)

So we have \( A = U_1(\mu_1; \alpha) \). Similarly we see that \( B = U_2(\mu_2; \alpha) \).

Thus \( A \times B = U(\mu; \alpha) \) is an ideal in \( Y \) for every \( \alpha \in [0, 1] \) \[\text{theorem (2.7)}\] Hence \( \mu \) is a fuzzy ideal of \( Y \).

Now we discuss fuzzy ideal for function algebra. For this we have to prove following results.

**Theorem (3.7)**- Let \((X; *, 1)\) be a CI–algebra and let \(F(X)\) be the class of all functions \(f : X \to X\). Let a binary operation \(\cdot \) be defined in \(F(X)\) as follows:

For \(f, g \in F(X)\) and \(x \in X\),

\[
(f \cdot g)(x) = f(x) * g(x).
\]

Then \((F(X); \cdot, 1^\ast)\) is a CI–algebra where \(1^\ast\) is defined as \(1^\ast(x) = 1\) for all \(x \in X\).

Here two functions \(f, g \in F(X)\) are equal iff \(f(x) = g(x)\) for all \(x \in X\).

**Proof**: Let \(f, g, h \in F(X)\). Then for \(x \in X\), we have

\[
(i) \quad (f \circ f)(x) = f(x) * f(x) = 1 = 1^\ast(x) \Rightarrow f \circ f = 1^\ast,
\]

\[
(ii) \quad (1^\ast \circ f)(x) = 1^\ast(x) \circ f(x) = f(x) \Rightarrow 1^\ast \circ f = f,
\]

\[
(iii) \quad (f \circ (g \circ h))(x) = f(x) * (g \circ h)(x)
\]

\[
= f(x) * (g(x) \cdot h(x))
\]

\[
= g(x) * (f(x) \cdot h(x))
\]

\[
= g(x) * (f \circ h)(x)
\]

\[
= (g \circ (f \circ h))(x).
\]

\[
\Rightarrow f \circ (g \circ h) = g \circ (f \circ h).
\]

This proves that \((F(X); \cdot, 1^\ast)\) is a CI–algebra.

**Theorem (3.8)**- Let \((X; *, 1)\) be a CI–algebra and let \((F(X); \cdot, 1^\ast)\) be CI–algebra considered in the above theorem. Then
I is an ideal of $X \iff F(I)$ is an ideal of $F(X)$.

**Proof:** Let $I$ be an ideal of $X$. For $f \in F(X)$ and $g \in F(I)$, we have $f(x) \in X$ and $g(x) \in I$ for all $x \in X$.

So $(f \circ g)(x) = f(x) * g(x) \in I$ for all $x \in X$.

This gives $f \circ g \in F(I)$.

Again, for $g, h \in F(I), f \in F(X)$ and $x \in X$, we have

$$((g \circ (h \circ f)) \circ f)(x) = (g \circ (h \circ f))(x) * f(x),$$

$$= (g(x) * (h(x) * f(x))) * f(x) \in I.$$

So $(g \circ (h \circ f)) \circ f \in F(I)$.

Hence $F(I)$ is an ideal in $F(X)$.

Conversely, suppose that $F(I)$ is an ideal of $F(X)$. Then

$f \in F(X)$ and $g \in F(I) \implies f \circ g \in F(I)$

$$\implies (f \circ g)(t) \in I \text{ for all } t \in X$$

$$\implies f(t) * g(t) \in I \text{ for all } t \in X. \quad (3.1)$$

Also $f \in F(X)$ and $g, h \in F(I)$

$$\implies (g \circ (h \circ f)) \circ f \in F(I)$$

$$\implies ((g \circ (h \circ f)) \circ f)(t) \in I \text{ for all } t \in X,$$

$$\implies (g(t) * (h(t)) * f(t)) * f(t) \in I \text{ for all } t \in X. \quad (3.2)$$

Let $x \in X$ and $a \in I$. We consider function $f_x$ and $f_a$ defined as $f_x(t) = x$ and $f_a(t) = a$ for all $t \in X$. \quad (3.3)

Now $f_x \in F(X)$ and $f_a \in F(I)$. So $f_x \circ f_a \in F(I)$.

This implies that $(f_x \circ f_a)(t) = f_x(t) \circ f_a(t) = x \circ a \in I$ for all $t \in X$ \quad [from (3.1)].

Again let $a, b \in I$ and $x \in X$. We consider functions $f_a$, $f_b$ and $f_x$ as defined by (3.3).

Then $f_x$, $f_a \in F(I)$ and $f_b \in F(X)$.

So $(f_a \circ (f_b \circ f_x)) \circ f_x \in F(I)$.

This gives $((f_a \circ (f_b \circ f_x)) \circ f_x)(t) = (a \circ (b \circ x)) \circ x \in I$ for all $t \in X$, \quad [from (3.2)].

Hence $I$ is an ideal of $X$.

**Definition (3.9):** Let $\mu$ be a fuzzy set defined on a finite $CI$–algebra $(X; \circ, 1)$. Let $(F(X); \circ, 1^*)$ be the $CI$–algebra considered in the theorem (3.7). We extend $\mu$ on $F(X)$ as
\[ \mathcal{P}(f) = \min \{ \mu(f(x)) : x \in X \} \]

We prove the following result.

**Lemma (3.10)**: If \( \mu(x) \leq \mu(1) \) for all \( x \in X \) then \( \mathcal{P}(f) \leq \mu(1) \) for all \( f \in F(X) \).

**Proof**: First of all we observe that \( \mathcal{P}(1^+) = \mu(1) \).

Since \( \mathcal{P}(1^+) = \min \{ \mu(1^+(x)) : x \in X \} \)

\[ = \mu(1). \]

Now \( \mathcal{P}(f) = \min \{ \mu(f(x)) : x \in X \} \)

\[ \leq \mu(1), \quad \text{since} \quad \mu(f(x)) \leq \mu(1) \quad \text{for all} \quad x \in X \]

\[ = \mathcal{P}(1^+). \]

**Lemma (3.11)**: \( F(\mathcal{P}(\mu ; \alpha)) = \mathcal{P}(\mu ; \alpha) \) for every \( \alpha \in [0, 1] \).

**Proof**: First of all we observe that for any \( \alpha \in [0, 1] \),

\( \mathcal{P}(\mu ; \alpha) \neq \phi \iff \mu(\alpha) \neq \phi \)

Let \( \mathcal{P}(\mu ; \alpha) \neq \phi \) and \( f \in F(\mathcal{P}(\mu ; \alpha)) \)

Then \( \mathcal{P}(f) \geq \alpha \). So \( \min \{ \mu(f(x)) : x \in X \} \geq \alpha \).

This implies that \( \mu(f(x)) \geq \alpha \) for some \( x \in X \),

i.e., \( f(x) \in U(\mu ; \alpha) \), and so \( \mu(\alpha) \neq \phi \).

Again let \( \mu(\alpha) \neq \phi \) and \( a \in U(\mu ; \alpha) \).

Then \( \mu(a) \geq \alpha \). If we choose \( f_a \in F(X) \) such that \( f_a(x) = a \) for all \( x \in X \). Then \( \mathcal{P}(f_a) = \min \{ \mu(f_a(x)) : x \in X \} = \mu(a) \geq \alpha \), i.e., \( f_a \in U(\mathcal{P}(\mu ; \alpha)) \) and so \( \mu(\alpha) \neq \phi \).

Now we see that \( f \in F(\mathcal{P}(\mu ; \alpha)) \iff f(x) \in U(\mu ; \alpha) \) for all \( x \in X \)

\[ \iff \mu(f(x)) \geq \alpha \quad \text{for all} \quad x \in X \]

\[ \iff \min \{ \mu(f(x)) : x \in X \} \geq \alpha \iff \mathcal{P}(f) \geq \alpha \iff f \in F(\mathcal{P}(\mu ; \alpha)). \]

This gives \( \mathcal{P}(\mu ; \alpha) = F(\mu ; \alpha) \).

**Corollary (3.12)**: If \( \mu(\alpha) \) is an ideal in \( X \) then \( \mathcal{P}(\mu ; \alpha) \) is an ideal in \( F(X) \).

**Proof**: This follows from above lemma and theorem (3.8).

**Theorem (3.13)**: If \( \mu \) is a fuzzy ideal of \( X \) then so is \( \mathcal{P}(\mu) \) on \( F(X) \).

**Proof**: Let \( \mu \) be a fuzzy ideal of \( X \). Then for every \( \alpha \in [0, 1] \)
U(µ; α) ≠ φ ⇒ U(µ; α) is an ideal in X

So F(U(µ; α)) is an ideal in F(X) by theorem (3.8).

Now if α ∈ [0, 1] and U(µ; α) ≠ φ then from discussion given in lemma (3.11) we see that, U(µ; α) = F(U(µ; α)) and so U(µ; α) is an ideal in F(X).

This proves that µ is a fuzzy ideal in F(X).

References