e-Supplement Submodules and e-Supplemented Modules

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Abstract

In this paper, we introduce the concepts e-supplement submodules and e-supplemented modules. We study these concepts and give some basic properties about them.

Key words:


1. Introduction

Let R be a commutative ring with 1 ≠ 0 and M is a unitary R-module. A submodule N of M is called essential (denoted by N ⊆ M), if for any nonzero submodule K of M, K ∩ N ≠ (0) [2]. And a proper submodule N of M is called small submodule (denoted by N ⊆ M), if N + K ∩ M, for any proper submodule K of M [2]. Recall that a submodule N of an R-module M is called e-small (denoted by N ⊆∗ M), if whenever N + K = M with K ⊆ M, then N ∩ K = (0) [5]. A submodule N is called closed submodule if for each nonzero submodule N of M (denoted by N ⊆ M), if N has no proper essential extension submodule in M, that is if N ⊆∗ K ⊆ M, then N = K [2]. The concept of supplement submodule appeared in [4], where a submodule N of an R-module M is called a supplement submodule in M if M = N + K for some submodule K of M and N is a minimal submodule with this property. In section 3 we introduce e-supplemented module, if every submodule of M has an e-supplement submodule.

2. e-Supplemented Submodules

In this section we present a new concept namely e-supplement submodule. We study this concept and give some of its basic properties.

Definition 2.1:

Let U ⊆ M. An essential submodule V of M is called e-supplement of U if U + V = M and V is a minimal essential in M with this property.

The following is a characterization of e-supplement submodule.

Theorem 2.2:

Let V ⊆∗ M, V is e-supplement of U if and only if M = U + V and U ∩ V ⊆∗ V.

Proof:

(⇒)

Let V ⊆∗ V. If V is e-supplement of U, so U + V = M. To prove U ∩ V ⊆∗ V.

Assume (U ∩ V) + K = V, for some K ⊆ V. Then M = U + (U ∩ V) + K, hence M = U + K. But K ⊆ V and V ⊆∗ M, so K ⊆∗ M. On the other hand, K ⊆ V and V is a minimal essential in M with the property U + V = M. Thus K = V and hence U ∩ V ⊆∗ V.

(⇐) suppose M = U + V and U ∩ V ⊆∗ V. To prove V is e-supplement of U, let K ⊆∗ M and K ⊆ V such that M = U + K, we must prove K = V. Since K ⊆∗ M and K ⊆ V, then K ⊆∗ V.

But V = M ∩ V = (U + K) ∩ V = K + (U ∩ V) by modular law. As (U ∩ V) ⊆∗ V and K ⊆∗ V, imply that K = V and hence V is e-supplement of U.

Remarks and Examples 2.3:

1- A supplement submodule need not be e-supplement, for example: < 3 > is a supplement of < 2 > in the Z-module Z, but < 3 > is not e-supplement, since < 3 > ⊆∗ Z.
2- e- supplement submodule need not be supplement, for example: \( < Z > \supseteq Z_d \leq_s Z_d \), \( < Z > \) is an e-supplement \( Z_d \), but \( < Z > \) is not supplement submodule of \( Z_d \).

3- \( Z\rho\) is a supplement of \( N \) (for each \( N\leq_s Z\rho\)). Also it is e-supplement

4- \( Z_d \) is a supplement of any \( N\leq Z_d \). Also it is e-supplement

**Proposition 2.4:**

Let \( A,N,K \) be submodules of an \( R \)-module \( M \) such that \( N \) is e-supplement of \( M \) and \( A \) is e-supplement of \( K \) in \( M \), then \( A \) is e-supplement of \( N \) in \( M \).

**Proof:**

Since \( N \) is e-supplement of \( A \), then \( N\leq M \) and \( A+N=M \). Also \( N \leq N \), and since \( A \) is e-supplement of \( K \) in \( M \), then \( A\leq M \). And, \( A+K=M \), \( A \) is minimal essential with the property \( A+K=M \).

To prove \( A \) is e-supplement of \( N \). Since \( A\leq M \), \( A+N=M \), so it is enough to show that \( A \) is minimal essential in \( M \) with the property \( A+N=M \). Let \( L \leq M \) and \( L \leq A \) such that \( L+N=M \). To prove \( L=A \). Since \( A=M\cap A=(L\cap A)+(N\cap A) \) by modular law, then \( M=+(N\cap A)=(L\cap A)+(L+K) \). But \( (N\cap A)\leq N \), then \( N\cap A \leq_i M \), also \( L\leq M \), implies \( L\leq K \leq M \). Hence \( M=L+K \). But \( A \) is e-supplement of \( K \) and \( L\leq M \), \( L \leq A \), so that \( L=A \). Thus \( A \) is e-supplement of \( N \).

**Proposition 2.5:**

Let \( A \), \( N \) be submodules of an \( R \)-module \( M \) such that \( N\leq A \). If \( N \) is e-supplement in \( M \), then \( N \) is e-supplement of \( A \).

**Proof:**

Since \( N \) is e-supplement in \( M \), then \( N\leq M \) and there exists \( K \leq M \) such that \( N+K=M \), \( N\cap K\leq N \), \( \ldots \) (1). Now \( A=M\cap A=(K+N)\cap A=N+(K\cap A) \) by modular law. Since \( N\cap K\leq M \), \( N\cap K\leq N \). Also \( N\leq A \). Thus \( N \) is e-supplement of \( K\cap A \) in \( A \).

**Proposition 2.6:**

Let \( M \) be an \( R \)-module, let \( A\leq N\leq M \) with \( N \) is an e-supplement in \( M \), then \( A \) is an e-supplement in \( N \) if and only if \( A \) is an e-supplement in \( M \).

**Proof:**

Since \( N \) is an e-supplement in \( M \), then \( N\leq M \) and there exists \( K \leq M \) such that \( N+K=M \) and \( N \) is minimal essential with this property. To prove that \( A \) is an e-supplement in \( M \). As \( A \) is an e-supplement in \( N \), so \( A\leq N \) and there exists \( L\leq N \) such that \( A+L=N \) and \( A \) is minimal essential submodule of \( N \) with this property. It follows that \( M=N+K=A+(L+K) \).

Since \( A\leq N \) and \( N\leq M \), we get \( A\leq_M \). Let \( B\leq L \), \( B\leq A \) such that \( B+L=N \), so \( B+L+K=M \). But \( B\leq M \), then \( B+L \leq M \). Also \( B+L \leq N \) and since \( N \) is minimal essential such that \( N+K=M \), so that \( B+L=N \), but \( B\leq A \) and \( A \) is a minimal essential submodule such that \( A+L=N \), so \( B=A \) and \( A \) is minimal essential with property \( A+(L+K)=M \); i.e. \( A \) is an e-supplement of \( L+K \).

(\( \Rightarrow \)) It follows by Proposition 2.5.

**Proposition 2.7:**

Let \( M_1 , M_2 \) be \( R \)-module, \( M=\bigoplus \) \( M_1 \). If \( A \) is an e-supplement of \( K_1 \) in \( M_1 \), \( B \) is an e-supplement of \( K_2 \) in \( M_2 \). Then \( A\bigoplus B \) is an e-supplement of \( K_1 \bigoplus K_2 \) in \( M \bigoplus M_2 \).

**Proof:**

\( A \) is an e-supplement of \( K_1 \) in \( M_1 \), then \( A\leq M_1 \) with \( A+K_1=M_1 \) and \( A\cap K_1\leq A \). \( B \) is an e-supplement of \( K_2 \) in \( M_2 \), then \( B\leq M_2 \) with \( B+K_2=M_2 \) and \( B\cap K_2\leq B \).

Now, \( M_1 \bigoplus M_2 =(A+K_1)\bigoplus (B+K_2) \). Also \( (A+K_1)\bigoplus (B+K_2) \leq (A\bigoplus B) \) \( \bigoplus \) \( (K_1\bigoplus K_2) \), \( \ldots \) (5, Proposition 2.5 (3)).

But \( A\leq M_1 \) and \( B\leq M_2 \), imply \( A\bigoplus B \leq M_1 \bigoplus M_2 \) \( \ldots \) (2, Proposition 1.3). Thus \( A\bigoplus B \) is an e-supplement of \( K_1 \bigoplus K_2 \).

**Proposition 2.8:**

Let \( M \) be an \( R \)-module, if \( A \) is an e-supplement of \( K \leq M \), let \( N\leq A \) and \( N \) is closed in \( M \), then \( A\bigoplus N \) is an e-supplement in \( M \).

**Proof:**
A is an e-supplement of K in M, so \( A \leq M \) and \( A + K = M \), \( A \cap K \ll_e A \). To prove that \( \frac{A}{N} \) is an e-supplement in \( \frac{M}{N} \). First since \( N \leq M \) and \( N \leq A \leq M \), then \( \frac{A}{N} \leq \frac{M}{N} \) by [2, Proposition 1.4, P.18]. Now, \( A + K = M \) implies \( \frac{A + K}{N} = \frac{M}{N} \), hence \( \frac{A + K}{N} = \frac{M}{N} \).

We claim that \( \frac{A}{N} \cap \frac{K + N}{N} \ll_e \frac{A}{N} \). Since \( \frac{A}{N} \cap \frac{K + N}{N} = \frac{A \cap (K + N)}{N} = \frac{N + (A \cap K)}{N} \) by modules law. Thus

\[
\frac{A}{N} \cap \frac{K + N}{N} = \frac{N + (A \cap K)}{N}.
\]

Let \( \frac{L}{N} \leq \frac{A}{N} \) such that \( \frac{N + (A \cap K)}{N} + \frac{L}{N} = \frac{A}{N} \). Then \( N + (A \cap K) + L = A \) and hence \( (A \cap K) + L = A \). But \( L \leq \frac{A}{N} \), implies \( L \subseteq A \) and since \( A \cap K \ll_e A \), then \( L = A \); that is \( \frac{L}{N} = \frac{A}{N} \). It follows that \( \frac{N}{N} \cap \frac{K + N}{N} \ll_e \frac{A}{N} \), so \( \frac{A}{N} \) is an e-supplement of \( \frac{K + N}{N} \).

**Remark 2.9:**

If \( A \) is an e-supplement of \( B \) and \( B \) is an e-supplement of \( C \), then it is not necessarily that \( A \) is an e-supplement of \( C \). For example, let \( V = \langle 2 \rangle \leq \langle 4 \rangle \). \( V \) is an e-supplement of \( \langle 4 \rangle \), and \( \langle 4 \rangle \) is an e-supplement of \( \langle 8 \rangle \). But \( V \) is not an e-supplement of \( \langle 8 \rangle \).

Recall that an \( R \)-module is called a multiplication module if for every submodule \( N \) of \( M \), there exists an ideal \( I \) of \( R \) such that \( IM = N \). Equivalently, \( M \) is a multiplication module if for every submodule \( N \) of \( M \), \( N = (N:IM) \). [1]

To prove the next result, we prove first the following lemma:

**Lemma 2.10:**

Let \( M \) be a finitely generated faithful multiplication \( R \)-module and let \( I \leq J \leq R \). If \( I \ll_e J \), then \( IM \ll_e JM \).

**Proof:**

Let \( K \subseteq JM \). As \( K \leq M \), \( K = LM \) for some \( L \leq R \), since \( M \) is a multiplication \( R \)-module. Assume that \( IM + K = JM \), so \( IM + LM = JM \). But \( M \) is a finitely generated faithful multiplication \( R \)-module, so \( I + L = J \). But we can show that \( L \leq J \) as follows, suppose \( T \in J \) and \( T \cap L = 0 \). Then \( (T \cap L)M = 0 \) and hence \( TM \cap LM = 0 \). But \( K = LM \leq JM \), so \( TM \leq JM \), hence \( T = 0 \) which implies \( L \leq J \). But \( I \ll_e J \), so \( L = J \). Thus \( K = LM = JM \) and \( IM \ll_e JM \).

**Proposition 2.11:**

Let \( M \) be a finitely generated faithful multiplication \( R \)-module and let \( N \leq M \). Then \( N \) is an e-supplement in \( M \) if and only if \( [N:M] \) is an e-supplement in \( R \).

**Proof:**

\( (\Rightarrow) \) If \( N \) is an e-supplement in \( M \), then \( N \leq M \) and there exists \( K \leq M \) such that \( N + K = M \) and \( N \cap K \ll_e N \). Since \( N \leq M \) and \( M \) is finitely generated faithful multiplicative \( R \)-module, then \( [N:M] \leq R \). Also \( N + K = M \), implies \( [N:M] + [K:M] = R \). To prove \( [N:M] \cap [K:M] \ll_e [N:M] \). First \( [N:M] \cap [K:M] = [N \cap K : M] \). Let \( I \leq [N:M] \). If \( [N \cap K : M] + I = [N:M] \), then \( [N \cap K : M] + I \subseteq [N:M : M] \), \( [N \cap K : M] + I \subseteq N \). But \( I \leq [N:M] \), then \( IM \subseteq N \) [by Lemma 2.10] and since \( N \cap K \ll_e N \), so \( IM = N = [N:M]M \). As \( M \) is a finitely generated faithful multiplication, we get \( I = [N:M] \).

Thus \( [N \cap K : M] \ll_e [N:M] \).

\( (\Leftarrow) \) If \( [N:M] \) is an e-supplement in \( R \), then \( [N:M] \leq R \) and there exists \( J \leq R \) such that \( [N:M] + J = R \), \( [N:M] \cap J \ll_e [N:M] \). Then \( N + JM = M \). But \( [N:M] \leq R \) implies \( N \leq [N:M] \) [1, Th. 2.13].

Thus \( N \) is an e-supplement in \( M \).

### 3. e-Supplemented Modules

In this section, we introduce a new class of module namely e-supplemented module, by using the concept of e-supplement submodule. This class of modules is a generalization of the class of supplemented modules.

**Definition 3.1:**

M is called an e-supplemented \( R \)-module if every submodule of \( M \) has an e-supplement submodule.

**Example 3.2:**

1. Consider \( Z_4 \) as \( Z \)-module,
   \( \langle 2 \rangle \) has an e-supplement in \( Z_4 \) which is \( Z_4 \), \( \langle 2 \rangle \) has an e-supplement in \( Z_4 \) which is \( Z_4 \).
   \( Z_4 \) has an e-supplement \( \langle 2 \rangle \). Thus \( Z_4 \) is an e-supplemented module

2. Consider \( Z_6 \) as \( Z \)-module, since each submodule of \( Z_6 \) has an e-supplement submodule which is \( Z_6 \). Thus \( Z_6 \) is an e-supplemented module.
3- Consider the Z-module Z , <Z> has an e-supplement Z . Let N< Z , then N=nZ , for some nÎZ, n>1 . Suppose mZ is an e-supplement of nZ , then nZ+mZ=Z . Thus g.c.d (m,n)=1 , so nZ + m'Z=Z and m'Z Î mZ=Z . If m≠±1 , then g.c.d (n,m')=1 , so nZ + m'Z=Z and m'Z Î mZ , so every proper submodule of Z has no e-supplement . Thus Z as Z-module is not e-supplemented Z-module.

**Remark 3.3:**
Let M be a semisimple R-module . Then M is e-supplement

**Proof:**
Since M is semisimple , then M is the only essential submodule of M and for each NÎM , N+M=M and N∩M=N∩M ; that is M is an e-supplement submodule of each submodule N of M

**Definition 3.4:**
1- Let N be a submodule of a module M . N is called e-weakly supplement of A in M if N≤M , N+A=M and N∩A≠N . M is called an e-weakly supplemented if every submodule of M has an e-weakly supplement.
2- M is called an e-amply supplement module if for any two submodule X and Y of M with M=X+Y , Y contains an e-supplement of X in M .

**Remark 3.5:**
For an R- module M. It is clear that
1- M is an amply e-supplemented module implies M is e-supplemented .

**Proposition 3.6:**
For an R-module M such that Rad,M(=0) . The following statements are equivalent :
1- M is a semisimple module .
2- M is an supplement module .
3- M is an e-weakly supplement module.

**Proof:**
(1) → (2) Since M is semisimple , then M the only essential in M and M+N=M , M∩N=N for any N≤M . But we can show that N≤N as follows. Let N+U=N U≤N . But N is semisimple , then the only essential in N is N it is self , so U=N and hence N≤N.
(2) → (3) It is clear ..
(3) → (1) Let N≤M , so there exists K≤M , K is e-weakly supplement , then N+K=M and N∩K≠K . But Rad,M(=0) , then N∩K(=0) . Thus N≤K .

**Proposition 3.7:**
Let M , M be R-modules and f:M → M be an R-epimorphism . If M is an e-supplemented module (e-supplemented or e-weakly supplemented) module then so is M .

**Proof:**
If M is an e-weakly supplemented module . Let X,Y ≤M , such that M/X=Y . Then f(X)+f(Y)=M (since f is onto) . But M is an e-weakly supplemented module , then f(X) contains C of f(Y) in M , if f(X)+C=M and f(X∩C)≤C . Now it is clear that X+ f(Y)=M . We claim that X∩f(Y) ≤ f(Y) , where f(C) ≤ Y . Since f(X)∩C≤C , then f(f(C))≤f(Y) . Hence X∩f(X∩C)≤f(Y) . Let y∈X∩f(Y) , then y=f(c) , for some c∈C , y=f(c)∈X , then X∩f(c)≤f(f(X∩C))≤f(Y) , thus f(Y) is an e-supplement of X in M .

The proof in similarly for M is e-supplemented module and M is an e-weakly supplemented module.

**References**