Some New Families of Face Edge Product Cordial Graphs

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ABSTRACT

In this paper, we investigate the face edge product cordial labeling of the alternative triangular snakes, \( K_{1,n} \otimes P_m \) and \((P_n \otimes K_1) \otimes P_m\). Also non face edge product cordial labeling of the graph DS\((P_n)\) is presented.

Keywords: Edge product cordial labeling, Alternate triangular snake, Face edge product cordial labeling, Face edge product cordial graph.

1. Introduction

Throughout this paper we consider only finite, planar, undirected and simple graphs. Let \( G \) be a graph with \( p \) vertices and \( q \) edges. For all terminologies and notations related to graph theory, we follow Harary [2]. For standard terminology and notations related to graph labeling, we refer to Gallian [1]. The concept of edge product cordial labeling of graphs is introduced by Vaidya et al. [5]. In [3], Lawrence et al. introduced the concept of face edge product cordial labeling of graphs and face edge product cordial labeling triangular snake is presented. Face edge product cordial labeling of some corona graphs are investigated by Muthaiyan et al. [4]. In [6], Vaidya et al. presented the product cordial labelings for alternate triangular snake graphs.

The brief summaries of definition which are necessary for the present investigation are provided below.

Definition : 1.1

The corona \( G_1 \otimes G_2 \) of two graphs \( G_1(p_1, q_1) \) and \( G_2(p_2, q_2) \) is defined as the graph obtained by taking one copy of \( G_1 \) and \( p_1 \) copies of \( G_2 \) and then joining the \( i^{th} \) vertex of \( G_1 \) to all the vertices in the \( i^{th} \) copy of \( G_2 \).

Definition : 1.2

For a graph \( G = (V(G), E(G)) \), an edge labeling function \( f : E(G) \rightarrow \{0, 1\} \) induces a vertex labeling function \( f^* : V(G) \rightarrow \{0, 1\} \) defined as \( f^*(v) = \prod f(e_i) \) for \( \{e_i \in E(G) : e_i \text{ is incident to } v\} \). Now denoting the number of vertices of \( G \) having label \( i \) under \( f^* \) as \( v_i \) and the number of edges of \( G \) having label \( i \) under \( f \) as \( e_i \). Then \( f \) is called edge product cordial labeling of graph \( G \) if \( |v_0 - v_1| \leq 1 \) and \( |e_0 - e_1| \leq 1 \). A graph \( G \) is called edge product cordial if it admits edge product cordial labeling.

Definition : 1.3

For a planar graph \( G \), the edge labeling function is defined as \( g : E(G) \rightarrow \{0, 1\} \) and \( g(e) \) is called the label of the edge \( e \) of \( G \) under \( g \), induced vertex labeling function \( g^* : V(G) \rightarrow \{0, 1\} \) is given as if \( e_1, e_2, \ldots, e_m \) are the edges incident to vertex \( v \), then \( g^*(v) = g(e_1)g(e_2)\ldots g(e_m) \) and induced face labeling function \( g^{**} : F(G) \rightarrow \{0, 1\} \)
is given as if $v_1, v_2, \ldots, v_n$ and $e_1, e_2, \ldots, e_m$ are the vertices and edges of face $f$ then $g^*(f) = g'(v_1)g'(v_2)\ldots g'(v_n) g(e_1)g(e_2)\ldots g(e_m)$. Let us denote $v_y(i)$ is the number of vertices of $G$ having label $i$ under $g$, $e_y(i)$ is the number of edges of $G$ having label $i$ under $g$ for $i = 1,2,\ldots, n$. That is every alternate edge of a path is replaced by $C_3$.

Definition : 1.6

Let $G = (V(G), E(G))$ be a graph with vertex set $V = S_1 \cup S_2 \cup \ldots \cup S_k \cup T$ where each $S_i$ is a set of vertices having at least two vertices of the same degree and $T = V \setminus S_i$. The degree splitting graph of $G$ denoted by $DS(G)$ is obtained from $G$ by adding vertices $w_1, w_2, w_3, \ldots, w_t$ and joining to each vertex of $S_i$ for $1 \leq i \leq t$.

In this paper, we investigate the face edge product cordial labeling of the alternative triangular snakes, $K_{1,n} \odot P_m$ and $(P_n \odot K_1) \odot P_m$. Also non face edge product cordial labeling of the planar graph $DS(P_n)$ is presented.

2. Main Theorems

Theorem 2.1

The graph alternative triangular snake $A(T_n)$ is face edge product cordial graph except $n \equiv 3 \pmod{4}$.

Proof.

Let $G$ be a alternative triangular snake $A(T_n)$. Let $v_1, v_2, \ldots, v_n$ and $e_1, e_2, \ldots, e_{n-1}$ be the vertices and edges of the path $P_n$.

Case 1 : $n \equiv 0 \pmod{4}$ and the first triangle start from $v_1$ and the last triangle ends with $v_n$.

To construct alternative triangular snake $A(T_n)$ from path $P_n$ by joining $v_i$ and $v_{i+1}$ alternatively with a new vertex $u_i$ by edges $e'_{2i-1} = v_{2i-1}u_i$ and $e'_{2i} = u_i v_{2i}$ for $i = 1, 2, \ldots, \frac{n}{2}$ and interior faces $f_i = v_{2i-1} u_i v_{2i}$ for $i = 1, 2, \ldots, \frac{n}{2}$.

Then $|V(G)| = \frac{3n}{2}$, $|E(G)| = 2n - 1$ and $|F(G)| = \frac{n}{2}$.

Define edge labeling $g : E(G) \rightarrow \{0,1\}$ as follows

\[
g(e_i) = 1, \quad \text{for } 1 \leq i \leq \frac{n}{2}
\]

\[
g(e_i) = 0, \quad \text{for } \frac{n+2}{2} \leq i \leq n - 1
\]

\[
g(e'_i) = 1, \quad \text{for } 1 \leq i \leq \frac{n}{2}
\]
g(e') = 0, \quad \text{for } \frac{n+2}{2} \leq i \leq n

In view of the above defined labeling pattern, we have

\[ e_g(1) = e_g(0) + 1 = n, \quad v_g(0) = v_g(1) = \frac{3n}{4} \quad \text{and} \quad f_g(0) = f_g(1) = \frac{n}{4}. \]

Therefore \( |e_g(0) - e_g(1)| \leq 1, \ |v_g(0) - v_g(1)| \leq 1 \) and \( |f_g(0) - f_g(1)| \leq 1 \)

Thus, the alternative triangular snake \( A(T_n) \) is face edge product cordial graph, when \( n \equiv 0 \) (mod 4) and the first triangle start from \( v_1 \) and the last triangle ends with \( v_n \).

**Case 2 :** \( n \equiv 0 \) (mod 4) and the first triangle start from \( v_2 \) and the last triangle ends with \( v_{n-1} \).

To construct alternative triangular snake \( A(T_n) \) from path \( P_n \) by joining \( v_i \) and \( v_{i+1} \) alternatively with a new vertex \( u_i \) by edges \( e'_{2i+1} = v_{2i}u_i \) and \( e'_{2i} = u_iv_{2i+1} \) for \( i = 1, 2, \ldots, \frac{n-2}{2} \) and interior faces \( f_i = v_{2i}u_i v_{2i+1} \) for \( i = 1, 2, \ldots, \frac{n-2}{2} \).

Then \( |V(G)| = \frac{3n-2}{2}, \ |E(G)| = 2n - 3 \) and \( |F(G)| = \frac{n-2}{2} \).

Define edge labeling \( g : E(G) \rightarrow \{0, 1\} \) as follows

\[ g(e_i) = 1, \quad \text{for } 1 \leq i \leq \frac{n}{2} \]

\[ g(e_i) = 0, \quad \text{for } \frac{n+2}{2} \leq i \leq n - 1 \]

\[ g(e'_i) = 1, \quad \text{for } 1 \leq i \leq \frac{n-2}{2} \]

\[ g(e'_i) = 0, \quad \text{for } \frac{n}{2} \leq i \leq n - 2 \]

In view of the above defined labeling pattern, we have

\[ e_g(1) = e_g(0) + 1 = n - 1, \quad v_g(0) = v_g(1) + 1 = \frac{3n}{4} \quad \text{and} \quad f_g(0) = f_g(1) + 1 = \frac{n}{4}. \]

Therefore \( |e_g(0) - e_g(1)| \leq 1, \ |v_g(0) - v_g(1)| \leq 1 \) and \( |f_g(0) - f_g(1)| \leq 1 \)

Thus, the alternative triangular snake \( A(T_n) \) is face edge product cordial graph, when \( n \equiv 0 \) (mod 4) and the first triangle start from \( v_2 \) and the last triangle ends with \( v_{n-1} \).

**Case 3 :** \( n \equiv 2 \) (mod 4) and the first triangle start from \( v_1 \) and the last triangle ends with \( v_{n} \).

Let path \( P_n \) having vertices \( v_1, v_2, \ldots, v_n \) and edges \( e_1, e_2, \ldots, e_{n-1} \).

To construct alternative triangular snake \( A(T_n) \) from path \( P_n \) by joining \( v_i \) and \( v_{i+1} \) alternatively with a new vertex \( u_i \) by edges \( e'_{2i+1} = v_{2i}u_i \) and \( e'_{2i} = u_iv_{2i+1} \) for \( i = 1, 2, \ldots, \frac{n}{2} \) and interior faces \( f_i = v_{2i-1}u_i v_{2i} \) for \( i = 1, 2, \ldots, \frac{n}{2} \).
Then \(|V(G)| = \frac{3n}{2}, |E(G)| = 2n - 1\) and \(|F(G)| = \frac{n}{2}\).

Define edge labeling \(g : E(G) \rightarrow \{0,1\}\) as follows

\[
\begin{align*}
g(e_i) &= 1, & \text{for } 1 \leq i \leq \frac{n}{2} \\
g(e_i) &= 0, & \text{for } \frac{n+2}{2} \leq i \leq n - 1 \\
g(e'_i) &= 1, & \text{for } 1 \leq i \leq \frac{n}{2} \\
g(e'_i) &= 0, & \text{for } \frac{n+2}{2} \leq i \leq n
\end{align*}
\]

In view of the above defined labeling pattern, we have

\[
e'_g(1) = e_g(0) + 1 = n - 1, \quad v'_g(0) = v_g(1) + 1 = \frac{3n+2}{4} \quad \text{and} \quad f'_g(0) = f_g(1) + 1 = \frac{n+2}{4}.
\]

Therefore \(|e'_g(0) - e'_g(1)| \leq 1, |v'_g(0) - v'_g(1)| \leq 1\) and \(|f'_g(0) - f'_g(1)| \leq 1\).

Thus, the alternative triangular snake \(A(T_n)\) is face edge product cordial graph, when \(n \equiv 2 \pmod{4}\) and the first triangle start from \(v_1\) and the last triangle ends with \(v_n\).

**Case 4 :** \(n \equiv 2 \pmod{4}\) and the first triangle start from \(v_2\) and the last triangle ends with \(v_{n-1}\).

To construct alternative triangular snake \(A(T_n)\) from path \(P_n\) by joining \(v_i\) and \(v_{i+1}\) alternatively with a new vertex \(u_i\) by edges \(e'_{2i+1} = v_{2i}u_i\) and \(e'_{2i} = u_iv_{2i+1}\) for \(i = 1,2,\ldots, \frac{n-2}{2}\) and interior faces \(f_i = v_{2i}u_i v_{2i+1}\) for \(i = 1,2,\ldots, \frac{n-2}{2}\).

Then \(|V(G)| = \frac{3n-2}{2}, |E(G)| = 2n - 3\) and \(|F(G)| = \frac{n-2}{2}\).

Define edge labeling \(g : E(G) \rightarrow \{0,1\}\) as follows

\[
\begin{align*}
g(e_i) &= 1, & \text{for } 1 \leq i \leq \frac{n}{2} \\
g(e_i) &= 0, & \text{for } \frac{n+2}{2} \leq i \leq n - 1 \\
g(e'_i) &= 1, & \text{for } 1 \leq i \leq \frac{n-2}{2} \\
g(e'_i) &= 0, & \text{for } \frac{n}{2} \leq i \leq n - 2
\end{align*}
\]

In view of the above defined labeling pattern, we have

\[
e'_g(1) = e_g(0) + 1 = n - 1, \quad v'_g(0) = v_g(1) = \frac{3n-2}{4} \quad \text{and} \quad f'_g(0) = f_g(1) = \frac{n-2}{4}.
\]

Therefore \(|e'_g(0) - e'_g(1)| \leq 1, |v'_g(0) - v'_g(1)| \leq 1\) and \(|f'_g(0) - f'_g(1)| \leq 1\).

Thus, the alternative triangular snake \(A(T_n)\) is face edge product cordial graph, when \(n \equiv 2 \pmod{4}\) and the first triangle start from \(v_2\) and the last triangle ends with \(v_{n-1}\).
Case 5: \(n \equiv 1 \pmod{4}\) and the first triangle start from \(v_1\) and the last triangle ends with \(v_{n-1}\).

To construct alternative triangular snake \(A(T_n)\) from path \(P_n\) by joining \(v_i\) and \(v_{i+1}\) alternatively with a new vertex \(u_i\) by edges \(e'_{2i+1} = v_{2i+1}u_i\) and \(e'_{2i} = u_i v_{2i}\) for \(i = 1, 2, \ldots, \frac{n-1}{2}\) and interior faces \(f_i = v_{2i-1} u_i v_{2i}\) for \(i = 1, 2, \ldots, \frac{n-1}{2}\).

Then \(|V(G)| = \frac{3n-1}{2}, |E(G)| = 2n - 2\) and \(|F(G)| = \frac{n-1}{2}\).

Define edge labeling \(g : E(G) \to \{0, 1\}\) as follows

\[
\begin{align*}
g(e_i) &= 1, & \text{for } 1 \leq i \leq \frac{n-1}{2} \\
g(e_i) &= 0, & \text{for } \frac{n+1}{2} \leq i \leq n - 1 \\
g(e'_i) &= 1, & \text{for } 1 \leq i \leq \frac{n-1}{2} \\
g(e'_i) &= 0, & \text{for } \frac{n+1}{2} \leq i \leq n - 1
\end{align*}
\]

In view of the above defined labeling pattern, we have

\[
e_g(1) = e_g(0) = n - 1, \ v_g(0) = v_g(1) + 1 = \frac{3n+1}{4} \quad \text{and} \quad f_g(0) = f_g(1) = \frac{n-1}{4}.
\]

Therefore \(|e_g(0) - e_g(1)| \leq 1, |v_g(0) - v_g(1)| \leq 1\) and \(|f_g(0) - f_g(1)| \leq 1\).

Thus, the alternative triangular snake \(A(T_n)\) is face edge product cordial graph, when \(n \equiv 1 \pmod{4}\) and the first triangle start from \(v_1\) and the last triangle ends with \(v_{n-1}\).

Case 6: \(n \equiv 1 \pmod{4}\) and the first triangle start from \(v_2\) and the last triangle ends with \(v_n\).

To construct alternative triangular snake \(A(T_n)\) from path \(P_n\) by joining \(v_i\) and \(v_{i+1}\) alternatively with a new vertex \(u_i\) by edges \(e'_{2i+1} = v_{2i+1}u_i\) and \(e'_{2i} = u_i v_{2i}\) for \(i = 1, 2, \ldots, \frac{n-1}{2}\) and interior faces \(f_i = v_{2i-1} u_i v_{2i+1}\) for \(i = 1, 2, \ldots, \frac{n-1}{2}\).

Then \(|V(G)| = \frac{3n-1}{2}, |E(G)| = 2n - 2\) and \(|F(G)| = \frac{n-1}{2}\).

Define edge labeling \(g : E(G) \to \{0, 1\}\) as follows

\[
\begin{align*}
g(e_i) &= 0, & \text{for } 1 \leq i \leq \frac{n-1}{2} \\
g(e_i) &= 1, & \text{for } \frac{n+1}{2} \leq i \leq n - 1 \\
g(e'_i) &= 0, & \text{for } 1 \leq i \leq \frac{n-1}{2} \\
g(e'_i) &= 1, & \text{for } \frac{n+1}{2} \leq i \leq n - 1
\end{align*}
\]
\( g(e_i) = 1 \), for \( \frac{n+1}{2} \leq i \leq n - 1 \)

In view of the above defined labeling pattern, we have

\[ e_g(1) = e_g(0) = n - 1, \quad v_g(0) = v_g(1) + 1 = \frac{3n+1}{4} \quad \text{and} \quad f_g(0) = f_g(1) = \frac{n-1}{2}. \]

Therefore \( |e_g(0) - e_g(1)| \leq 1, \quad |v_g(0) - v_g(1)| \leq 1 \) and \( |f_g(0) - f_g(1)| \leq 1 \)

Thus, the alternative triangular snake \( A(T_n) \) is face edge product cordial graph, when \( n \equiv 1(\text{mod } 4) \) and the first triangle start from \( v_1 \) and the last triangle ends with \( v_n \).

**Case 7**: \( n \equiv 3 \pmod{4} \) and the first triangle start from \( v_1 \) and the last triangle ends with \( v_{n-1} \).

In order to satisfy the edge condition for \( G \), it is essential to assign label 0 and 1 to exactly \( n-1 \) edges.

Any pattern of edge labeling which satisfies the edge condition will induce vertex labels for \( \frac{3n-1}{2} \) number of vertices in such a way that \( |v_g(0) - v_g(1)| \geq 2 \).

Therefore the vertex condition for \( G \) is violated. Thus the graph \( G \) under this consideration is not a face edge product cordial graph.

Hence, the alternative triangular snake \( A(T_n) \) is not face edge product cordial graph, when \( n \equiv 3(\text{mod } 4) \) and the first triangle start from \( v_1 \) and the last triangle ends with \( v_{n-1} \).

**Case 8**: \( n \equiv 3(\text{mod } 4) \) and the first triangle start from \( v_2 \) and the last triangle ends with \( v_{n-1} \).

In order to satisfy the edge condition for \( G \), it is essential to assign label 0 and 1 to exactly \( n-1 \) edges.

Any pattern of edge labeling which satisfies the edge condition will induce vertex labels for \( \frac{3n-1}{2} \) number of vertices in such a way that \( |v_g(0) - v_g(1)| \geq 2 \).

Therefore the vertex condition for \( G \) is violated. Thus the graph \( G \) under this consideration is not a face edge product cordial graph.

Hence, the alternative triangular snake \( A(T_n) \) is not face edge product cordial graph, when \( n \equiv 3 \pmod{4} \) and the first triangle start from \( v_2 \) and the last triangle ends with \( v_{n-1} \).

Therefore, the alternative triangular snake \( A(T_n) \) is face edge product cordial graph except \( n \equiv 3(\text{mod } 4) \).

**Example : 2.1**

(i). The alternative triangular snake \( A(T_6) \) with first triangle start from first vertex and its face edge product cordial labeling are shown in Figure 2.1(a).
(ii). The alternative triangular snake $A(T_0)$ with first triangle start from second vertex and its face edge product cordial labeling are shown in Figure 2.1(b).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.1a}
\caption{Figure 2.1(a)}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.1b}
\caption{Figure 2.1(b)}
\end{figure}

**Theorem 2.2**

$K_{1,n} \odot P_m$ is face edge product cordial graph except $n$ is even and $m > 2$.

**Proof.**

Let $u_1, u_2, \ldots, u_{n+1}$ and $e_1, e_2, \ldots, e_n$ be the vertices and edges of $K_{1,n}$.

Let $G$ be the graph $K_{1,n} \odot P_m$.

The vertex set $V(G) = \{u_i, v_{ij} : 1 \leq i \leq n+1, 1 \leq j \leq m\}$, edge set $E(G) = \{e_i, e_{jk} : 1 \leq i \leq n, 1 \leq j \leq n+1, 1 \leq k \leq 2m-1\}$ and interior face set $F(G) = \{f_i : 1 \leq i \leq (n+1)(m-1)\}$ of $G$, where $e_i = u_i u_{i+1}$ for $1 \leq i \leq n$, $e_{jk} = u_j v_k$ for $1 \leq j \leq n+1$ and $1 \leq k \leq m$, $e_{(m+1)k} = v_k v_{(k+1)}$, for $1 \leq j \leq n+1$ and $1 \leq k \leq m-1$, $f_i = u_i v_{(i-1)m} u_i$ for $1 \leq i \leq n+1$ and $1 \leq k \leq m-1$.

Then $|V(G)| = (n+1)(m+1)$, $|E(G)| = 2(n+1)m – 1$ and $|F(G)| = (n+1)(m-1)$.

Define edge labeling $g : E(G) \rightarrow \{0, 1\}$ as follows

**Case 1 :** $n$ is odd

\[
g(e_i) = 1, \quad \text{for } 1 \leq i \leq \frac{n+1}{2} \]

\[
g(e_i) = 0, \quad \text{for } \frac{n+3}{2} \leq i \leq n \]

\[
g(e_{jk}) = 1, \quad \text{for } 1 \leq j \leq \frac{n+1}{2} \text{ and } 1 \leq k \leq 2m-1 \]

\[
g(e_{jk}) = 0, \quad \text{for } \frac{n+3}{2} \leq j \leq n+1 \text{ and } 1 \leq k \leq 2m-1 \]

In view of the above defined labeling pattern we have

\[
e_g(1) = e_g(0) + 1 = (n+1)m, \quad v_g(0) = v_g(1) = \frac{(n+1)(m+1)}{2} \quad \text{and} \quad f_g(0) = f_g(1) = \frac{(n+1)(m-1)}{2}.\]

Then $|v_g(0) - v_g(1)| \leq 1$, $|e_g(0) - e_g(1)| \leq 1$ and $|f_g(0) - f_g(1)| \leq 1$.

Therefore, $K_{1,n} \odot P_m$ is face edge product cordial graph for $n$ is odd.

**Case 2 :** $n$ is even

**Sub case 2.1 :** $n$ is even and $m = 2$.
g(e_i) = 1, \quad \text{for } 1 \leq i \leq \frac{n}{2}

g(e_i) = 0, \quad \text{for } \frac{n+3}{2} \leq i \leq n

g(e_j) = 1, \quad \text{for } 1 \leq j \leq \frac{n+1}{2} \quad \text{and} \quad 1 \leq k \leq 2m - 1

g(e_j) = 0, \quad \text{for } \frac{n+3}{2} \leq j \leq n+1 \quad \text{and} \quad 1 \leq k \leq 2m - 1

g(e_{j+k}) = 1, \quad \text{for } j = n+1 \quad \text{and} \quad k = 1,3,

g(e_{j+k}) = 0, \quad \text{for } j = n+1 \quad \text{and} \quad k = 2.

In view of the above defined labeling pattern we have

e_{g(1)} = e_{g(0)+1} = 2n+2, \quad v_{g(1)}(0) = v_{g(1)+1} = \frac{3n+4}{2} \quad \text{and} \quad f_{g(1)}(0) = f_{g(1)+1} = \frac{n+2}{2}.

Then |v_{g(0)} - v_{g(1)}| \leq 1, \quad |e_{g(0)} - e_{g(1)}| \leq 1 \quad \text{and} \quad |f_{g(0)} - f_{g(1)}| \leq 1.

Therefore, \(K_{1,n} \circ P_m\) is face edge product cordial graph for \(n\) is even and \(m = 2\).

**Sub case 2.2 :** \(n\) is even and \(m > 2\).

In order to satisfy the edge condition for \(G\), it is essential to assign label 1 to atmost \((n+1)m\) edges out of \(2(n+1)m - 1\) edges. Assigning any pattern of edge labels which satisfying the edge condition will induce face labels for \((n+1)(m-1)\) number of faces in such a way that \(|f_{g(0)} - f_{g(1)}| \geq m - 1\), that is face condition for \(G\) is violated. Thus the graph \(G\) under consideration is not a face edge product cordial graph when \(n\) is even and \(m > 2\). Therefore, the graph \(K_{1,n} \circ P_m\) is not a face edge product cordial graph for \(n\) is even and \(m > 2\).

Hence, the graph \(K_{1,n} \circ P_m\) is face edge product cordial graph except \(n\) is even and \(m > 2\).

**Example : 2.2**

The graph \(K_{1,5} \circ P_3\) and its face edge product cordial labeling is given in figure 2.2.

**Theorem : 2.3**

\((P_n \circ K_1) \circ P_m\) is face edge product cordial graph.

**Proof.**

Let \(u_1, u_2, \ldots, u_{2n}\) and \(e_1, e_2, \ldots, e_{2m-1}\) be the vertices and edges of the comb graph \(P_n \circ K_1\).

Let \(G\) be the graph \((P_n \circ K_1) \circ P_m\).
The vertex set $V(G) = \{u_i, v_{ij} : 1 \leq i \leq 2n, 1 \leq j \leq m\}$, edge set $E(G) = \{e_i, e_{ik} : 1 \leq i \leq 2n–1, 1 \leq j \leq 2n and 1 \leq k \leq 2m–1\}$ and interior face set $F(G) = \{f_i : 1 \leq i \leq 2n(m–1)\}$, where $e_i = u_iu_{i+1}$ for $1 \leq i \leq n–1$, $e_{(n–1)n} = u_1u_{(n–1)n}$ for $1 \leq i \leq n$, $e_{ik} = u_iV_{ij}$ for $1 \leq j \leq 2n$ and $1 \leq k \leq m$, $e_{(n–1)k} = v_{jk}V_{(jk+1)}$ for $1 \leq j \leq 2n$ and $1 \leq k \leq m–1$, $f_i = u_iV_{ik}$ for $1 \leq i \leq 2n$ and $1 \leq k \leq m–1$.

Then $|V(G)| = 2n(m+1)$, $|E(G)| = 4nm–1$ and $|F(G)| = 2n(m–1)$.

Define edge labeling $g : E(G) \rightarrow \{0, 1\}$ as follows

**Case 1:** $n$ is odd

\[
g(e_i) = 1, \quad \text{for } 1 \leq i \leq \frac{n–1}{2}
\]

\[
g(e_i) = 0, \quad \text{for } \frac{n+1}{2} \leq i \leq n–1
\]

\[
g(e_{n–1n}) = 1, \quad \text{for } 1 \leq i \leq \frac{n+1}{2}
\]

\[
g(e_{n–1n}) = 0, \quad \text{for } \frac{n+3}{2} \leq i \leq n
\]

\[
g(e_{jk}) = 1, \quad \text{for } 1 \leq j \leq \frac{n–1}{2} \text{ and } 1 \leq k \leq 2m–1
\]

\[
g(e_{jk}) = 0, \quad \text{for } \frac{n+1}{2} \leq j \leq n \text{ and } 1 \leq k \leq 2m–1
\]

\[
g(e_{(n–1)k}) = 1, \quad \text{for } 1 \leq j \leq \frac{n+1}{2} \text{ and } 1 \leq k \leq 2m–1
\]

\[
g(e_{(n–1)k}) = 0, \quad \text{for } \frac{n+3}{2} \leq j \leq n \text{ and } 1 \leq k \leq 2m–1
\]

In view of the above defined labeling pattern we have

\[e_{jk}^{(1)} = e_{jk}^{(0)} + 1 = 2nm, \quad v_{jk}^{(0)} = v_{jk}^{(1)} = n(m+1) \text{ and } f_{jk}^{(0)} = f_{jk}^{(1)} = n(m–1).
\]

Then $|v_{jk}^{(0)} – v_{jk}^{(1)}| \leq 1, \quad |e_{jk}^{(0)} – e_{jk}^{(1)}| \leq 1$ and $|f_{jk}^{(0)} – f_{jk}^{(1)}| \leq 1$

**Case 2:** $n$ is even

\[
g(e_i) = 1, \quad \text{for } 1 \leq i \leq \frac{n}{2}
\]

\[
g(e_i) = 0, \quad \text{for } \frac{n+2}{2} \leq i \leq n–1
\]

\[
g(e_{n–1n}) = 1, \quad \text{for } 1 \leq i \leq \frac{n}{2}
\]

\[
g(e_{n–1n}) = 0, \quad \text{for } \frac{n+2}{2} \leq i \leq n
\]

\[
g(e_{jk}) = 1, \quad \text{for } 1 \leq j \leq \frac{n}{2} \text{ and } 1 \leq k \leq 2m–1
\]

\[
g(e_{jk}) = 0, \quad \text{for } \frac{n+2}{2} \leq j \leq n \text{ and } 1 \leq k \leq 2m–1
\]

\[
g(e_{(n–1)n}) = 1, \quad \text{for } 1 \leq j \leq \frac{n}{2} \text{ and } 1 \leq k \leq 2m–1
\]
\[ g(e_{nm+k}) = 0, \quad \text{for } \frac{n+2}{2} \leq j \leq n \text{ and } 1 \leq k \leq 2m-1 \]

In view of the above defined labeling pattern we have
\[ e_g(1) = e_g(0) + 1 = 2nm, \quad v_g(0) = v_g(1) = n(m+1) \text{ and } f_g(0) = f_g(1) = n(m-1). \]

Then \(|v_g(0) - v_g(1)| \leq 1, \quad |e_g(0) - e_g(1)| \leq 1 \text{ and } |f_g(0) - f_g(1)| \leq 1\]

Therefore, the graph \((P_n \circledast K_1) \circledast P_m\) is face edge product cordial graph.

**Example : 2.3**

The graph \((P_3 \circledast K_1) \circledast P_3\) and its face edge product cordial labeling is given in figure 2.3.

![Figure 2.3](image_url)

**Theorem : 2.4**

The graph \(DS(P_n)\) is not face edge product cordial graph for \(n \geq 3\).

**Proof.**

Let \(G\) be the graph \(DS(P_n)\).

Let \(v_1, v_2, \ldots, v_n\) be the vertices of \(P_n\).

Now in order to obtain \(DS(P_n)\) from \(P_n\), add the vertices are \(w_1\) and \(w_2\) and add edges are \(v_1w_1, v_nw_1\) and \(v_iw_2\) for \(i = 2, 3, \ldots, n-1\).

Then \(|V(G)| = n+2, \quad E(G) = 2n-1 \text{ and } |F(G)| = n-1.\)

To define \(g : V(G) \rightarrow \{0, 1\}\) as follows

**Case 1:** \(n = 3\)

The graph \(DS(P_3) \equiv C_4\).

\(C_4\) is not an edge product cordial graph.

Therefore, \(DS(P_3)\) is not face edge product cordial graph.

**Case 2:** \(n \geq 4\)

In order to satisfy the edge condition for face edge product cordial graph it is essential to assign label 0 to

\[ \left\lfloor \frac{2n-1}{2} \right\rfloor \]

edges out of \(2n - 1\) edges. The edges with label 0 will give rise all the \(n-2\) faces with label 0.

Therefore, \(|f_g(0) - f_g(1)| > 2.\)

Hence, the graph \(DS(P_n)\) is not face edge product cordial graph for \(n \geq 3.\)
References:


