Signed Product Cordial labeling in duplicate graphs of Bistar, Double Star and Triangular Ladder Graph

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Abstract -In this paper, we prove that the duplicate graph of triangular ladder $DG(TL_m)$, $m \geq 2$, extended duplicate graph of bistar $EDG(B_{m,m})$, $m \geq 2$ and extended duplicate graph of double star $EDG(DS_{m,m})$, $m \geq 2$ admits signed product cordial labeling.

Keywords - Graph labeling, duplicate graph, triangular ladder, bistar, double star, signed product cordial labeling.

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I. INTRODUCTION

The concept of graph labeling was introduced by Rosa in 1967 [6]. A graph labeling is an assignment of integers to the vertices or edges or both subject to certain condition(s). If the domain of the mapping is the set of vertices (or edges), then the labeling is called a vertex labeling (or an edge labeling). In the intervening years various labeling of graphs have been investigated in over 2000 papers [4]. The concept of cordial labeling was introduced by I. Cahit[2]. The concept of duplicate graph was introduced by E. Sampathkumar and he proved many results on it [7]. The concept of signed product cordial labeling was introduced by J. BaskarBabjee and he proved that many graphs admit signed product cordial labeling. K. Thirusangu, P.P. Ulaganathan and B. Selvam, have proved that the duplicate graph of a path graph $P_m$ is Cordial [8].K. Thirusangu, P.P. Ulaganathan and P. Vijayakumar have proved that the duplicate graph of Ladder graph $L_m$, $m \geq 2$, is cordial, total cordial and prime cordial[9].

II. PRELIMINARIES

In this section, we give the basic notions relevant to this paper.

Definition 2.1: A graph labeling is an assignment of integers to the vertices or edges or both subject to certain condition(s). If the domain of the mapping is the set of vertices (or edges), then the labeling is called a vertex labeling (or an edge labeling).

Definition 2.3: Let $G(V,E)$ be a simple graph. A duplicate graph of $G$ is $DG = (V_1 \cup V_2, E_1 \cup E_2)$ where the vertex set $V_1 = V \cup V'$ and $V \cap V' = \emptyset$ and $f : V \rightarrow V'$ is bijective (for $v \in V$, we write $f(v) = v'$) and the edge set $E_1$ of $DG$ is defined as: The edge $uv$ in $E$ is in $E_1$ if and only if both $uv'$ and $u'v$ are edges in $E_1$.

Definition 2.4: A vertex labeling of graph $G$ $f : V(G) \rightarrow \{-1, 1\}$ with induced edge labeling $f^* : E(G) \rightarrow \{-1, 1\}$ defined by $f^* (uv) = f(u)f(v)$ is called a signed product cordial labeling if $|v_f(-1) - v_f(1)| \leq 1$ and $|e_f(-1) - e_f(1)| \leq 1$, where $v_f(-1)$ is the number of vertices labeled with -1, $v_f(1)$ is the number of vertices labeled with 1, $e_f(-1)$ is the number of edges labeled with -1 and $e_f(1)$ is the number of edges labeled with 1.

Definition 2.5: The ladder graph $L_m$ is a planar undirected graph with $2m$ vertices and $3m - 2$ edges. It is obtained as the cartesian product of two path graphs, one of which has only one edge: $L_m = P_m \times P_1$, where $m$ is the number of rungs in the ladder.

Definition 2.6: The triangular ladder graph $TL_m$ is obtained from the ladder graph by joining $v_i$ and $v_{i+2}$, $1 \leq i \leq m + 2$.

Definition 2.7: The Star graph $S_m$ is a complete bipartite graph $K_{1,m}$, where $m$ represents the number of vertices and $S_m$ has $(m - 1)$ edges.

Definition 2.8: Double star $DS_{m,m}$ is a tree $K_{1,m}$ obtained from the star $K_{1,m}$ by adding a new pendent.
edge of the existing m pendant vertices. It has 2m + 1 vertices and 2m edges. [11].

Definition 2.9: Bistar $B_{2m}$ is a graph obtained from $K_2$ by joining m pendant edges to each end of $K_2$. The edge $K_2$ is called the central edge of $B_{2m}$ and the vertices of $K_2$ are called the central vertices of $B_{2m}$. [5].

Definition 2.10: The extended duplicate graph of bistar denoted by $EDG(B_{2m})$, is obtained from the duplicate graph of bistar by joining $v_1$ and $v'_1$.

Definition 2.11: The extended duplicate graph of double star denoted by $EDG(DS_{2m})$, is obtained from the duplicate graph of double star by joining $v_1$ and $v'_1$.

III. MAIN RESULTS

Algorithm 3.1: Construction of duplicate graph of triangular ladder $DG(TL_m)$.

$V \leftarrow \{v_1, v_2, v_3, ..., v_{2m}, v'_1, v'_2, ..., v'_{2m}\}$
$E \leftarrow \{e_1, e_2, e_3, ..., e_{4m-3}, e'_1, e'_2, e'_3, ..., e'_{4m-3}\}$
for $1 \leq k \leq m - 1$
$v_{2k-1}v_{2m} \leftarrow e_{4k-3}, v_{2k-1}v'_{2k+1} \leftarrow e_{4k-2},$
$v_{2k-1}v'_{2k} \leftarrow e_{4k-1}, v_{2k}v'_{2k+2} \leftarrow e_{4k},$
$v'_{2k-1}v_{2k} \leftarrow e'_1, v'_{2k-1}v_{2k+1} \leftarrow e'_{4k-2},$
$v'_{2k-1}v'_{2k+2} \leftarrow e'_1, v'_2v'_{2k+2} \leftarrow e'_4.$

for $k = m$
$v_{2k-1}v'_{2k} \leftarrow e_{4k-3}, v'_{2k-1}v_{2k} \leftarrow e'_{4k-3}.$

Illustration

Remark: The Duplicate graph of Triangular ladder $DG(TL_m)$ has $2m$ vertices and $8m - 6$ edges.

Algorithm3.2: Assignment of labels to vertices

$V \leftarrow \{v_1, v_2, v_3, ..., v_{2m}, v'_1, v'_2, ..., v'_{2m}\}$
$E \leftarrow \{e_1, e_2, e_3, ..., e_{4m-3}, e'_1, e'_2, e'_3, ..., e'_{4m-3}\}$
for $1 \leq k \leq m$
$v_{2k-1} \leftarrow 1, v_{2k} \leftarrow -1;$
for $1 \leq k \leq m - 1$
$v'_{2k-1} \leftarrow -1, v'_{2k} \leftarrow 1;$
for $k = m$
$v'_{2k-1} \leftarrow 1, v'_{2k} \leftarrow -1.$

Theorem 3.1: The duplicate graph of triangular ladder $DG(TL_m)$, $m \geq 2$, admits signed product cordial labeling.

Proof: Let $V = \{v_1, v_2, ..., v_{2m}, v'_1, v'_2, ..., v'_{2m}\}$ and $E = \{e_1, e_2, ..., e_{4m-3}, e'_1, e'_2, ..., e'_{4m-3}\}$ be the set of vertices and edges of the duplicate graph of triangular ladder.

Using the algorithm 3.2, each of the $2m$ vertices receive label 1 and – 1. Using the induced function $f^*$ defined by

$f^*(uv) = f(u)f(v)$

The $4m - 3$ edges namely $e_1, e_3, e_5, e_7, e_9, ...,$ $e_{4m-9}, e_{4m-7}, e_{4m-5},$ $e_{4m-3}, e'_1, e'_3, e'_5, ...,$ $e'_{4m-9}, e'_{4m-7}, e'_{4m-5},$ receive label 1 and the $4m - 3$ edges namely $e_2, e_4, e_6, e_8, ...,$ $e_{4m-8}, e_{4m-6}, e_{4m-4}, e'_2, e'_4, e'_6, ...,$ $e'_{4m-8}, e'_{4m-6}, e'_{4m-4},$ receive label – 1 respectively. Thus the number of edges labeled with 1 and – 1 differ at most by one. Hence the duplicate graph of triangular ladder $DG(TL_m)$, $m \geq 2$, admits signed product cordial labeling.

Fig. 1 Example of Triangular ladder $(TL_3)$ and its duplicate graph $EDG(TL_3)$
Algorithm 3.3: (Construction of extended duplicate graph of Bistar \(DG(B_{m,m})\)).

\[
\begin{aligned}
V & \leftarrow \{v_1, v_2, v_3, \ldots, v_{2m+2}, v_i, v'_i, \ldots, v'_{2m+2}\} \\
E & \leftarrow \{e_1, e_2, e_3, \ldots, e_{2m+2}, e'_1, e'_2, \ldots, e'_{2m+1}\}
\end{aligned}
\]

\[
\begin{aligned}
&\text{fix } v_1, v'_1 \leftarrow e_{2m+2} \\
&\text{for } 2 \leq k \leq m + 2 \\
&\quad v_1v'_k \leftarrow e_{k-1}, v'_1v_k \leftarrow e'_{k-1}
\end{aligned}
\]

\[
\begin{aligned}
&\text{for } m + 2 \leq k \leq 2m + 1 \\
&\quad v_{m+2}v'_{k+1} \leftarrow e_k, v'_{m+2}v_{k+1} \leftarrow e'_k.
\end{aligned}
\]

Illustration:

Fig.3 Example of bistar \((B_{3,3})\) and \(EDG(B_{3,3})\)

Remark: The extended duplicate graph of Bistar \(EDG(B_{m,m})\) has \(4m + 4\) vertices and \(4m+3\) edges.

Algorithm 3.4: Assignment of labels to vertices

\[
\begin{aligned}
V & \leftarrow \{v_1, v_2, v_3, \ldots, v_{2m+2}, v'_1, v'_2, \ldots, v'_{2m+2}\} \\
E & \leftarrow \{e_1, e_2, e_3, \ldots, e_{2m+2}, e'_1, e'_2, \ldots, e'_{2m+1}\}
\end{aligned}
\]

\[
\begin{aligned}
&\text{for } 1 \leq k \leq m + 1 \\
&\quad v_{2k-1} \leftarrow 1, v_{2k} \leftarrow -1; \\
&\quad v'_{2k-1} \leftarrow -1, v'_{2k} \leftarrow 1.
\end{aligned}
\]

Theorem 3.2: The extended duplicate graph of the bistar \(EDG(B_{m,m})\), \(m \geq 2\), admits signed product cordial labeling.

Proof:

Let \(V = \{v_1, v_2, \ldots, v_{2m+2}, v'_1, v'_2, \ldots, v'_{2m+2}\}\) and \(E = \{e_1, e_2, \ldots, e_{2m+2}, e'_1, e'_2, \ldots, e'_{2m+1}\}\) be the set of vertices and edges of the \(EDG(B_{m,m})\).

Case (i): When \(m\) is odd

Using the algorithm 3.4 each of the \(2m + 2\) vertices receive label 1 and -1. Using the induced function defined in theorem 3.1, the \(2m + 2\) edges namely \(e_1, e_2, e_3, \ldots, e_{2m+1}, e'_1, e'_2, \ldots, e'_{2m+1}\) receive label 1 and the \(2m + 1\) edges namely \(e_2, e_4, e_6, \ldots, e_{2m-2}, e_{2m}, e_{2m+2}, e'_2, e'_4, e'_6, \ldots, e'_{2m-2}, e'_{2m}\) receive label -1. Thus the number of edges labeled with 1 and -1 differ at most by one.

Case (ii): When \(m\) is even

Using the algorithm 3.4 each of the \(2m + 2\) vertices receive label 1 and -1. Using the induced function defined in theorem 3.1, the \(2m + 2\) edges namely \(e_1, e_2, e_3, e_4, \ldots, e_{2m+1}, e_{m+1}, e_{m+3}, e_{m+5}, \ldots, e_{2m}, e'_1, e'_2, \ldots, e'_{m+1}, e'_{m+3}, e'_{m+5}, \ldots, e'_{2m}, e'_{2m+1}\) receive label 1 and the \(2m + 1\) edges namely \(e_2, e_4, e_6, \ldots, e_{2m-2}, e_{2m}, e_{2m+2}, e'_2, e'_4, e'_6, \ldots, e'_{2m-2}, e'_{2m}, e'_{2m+1}\) receive label -1. Thus the number of edges labeled with 1 and -1 differ at most by one.

Hence the extended duplicate graph of the bistar \(EDG(B_{m,m})\), \(m \geq 2\), admits signed product cordial labeling.

Illustration:

Fig 4 Example of signed product cordial labeling in \(EDG(B_{3,3})\)

Algorithm 3.5: (Construction of extended duplicate graph of double star \(DS_{m,m}\))
Illustration:

**Algorithm 3.6:** (Assignment of labels to vertices)

**Case (i): when** $m \equiv 0 \pmod{4}$

1. $v_{m+1} \leftarrow 1, v'_{m+1} \leftarrow -1$

2. For $1 \leq k \leq \frac{m}{4}$:
   - $v_{4k-3} \leftarrow 1, v_{4k-2} \leftarrow 1, v_{4k-1} \leftarrow -1, v_{4k} \leftarrow -1$
   - $v'_{4k-3} \leftarrow -1, v'_{4k-2} \leftarrow 1, v'_{4k-1} \leftarrow 1, v'_{4k} \leftarrow 1$

3. For $k = \frac{m+1}{2}$:
   - $v_{4k} \leftarrow 1, v_{4k+1} \leftarrow 1, v_{4k+2} \leftarrow 1, v_{4k+3} \leftarrow -1$

4. For $\frac{m+2}{4} \leq k \leq \frac{m-2}{4}$:
   - $v_{4k} \leftarrow 1, v_{4k+1} \leftarrow 1, v_{4k+2} \leftarrow -1, v_{4k+3} \leftarrow -1$

5. For $\frac{m-2}{4} \leq k \leq \frac{m-1}{2}$:
   - $v_{4k} \leftarrow -1, v_{4k+1} \leftarrow 1, v_{4k+2} \leftarrow 1, v_{4k+3} \leftarrow -1$

**Case (ii): when** $m \equiv 1 \pmod{4}$

1. $v_{4k-2} \leftarrow 1, v_{4k-1} \leftarrow -1, v_{4k} \leftarrow -1$

2. For $1 \leq k \leq \frac{m-1}{2}$:
   - $v_{4k-3} \leftarrow 1, v_{4k-2} \leftarrow 1, v_{4k-1} \leftarrow -1, v_{4k} \leftarrow -1$
   - $v'_{4k-3} \leftarrow -1, v'_{4k-2} \leftarrow 1, v'_{4k-1} \leftarrow 1, v'_{4k} \leftarrow 1$

3. For $k = \frac{m+1}{2}$:
   - $v_{4k} \leftarrow 1, v_{4k+1} \leftarrow 1, v_{4k+2} \leftarrow 1, v_{4k+3} \leftarrow -1$

4. For $\frac{m-1}{2} \leq k \leq \frac{m-1}{2}$:
   - $v_{4k} \leftarrow -1, v_{4k+1} \leftarrow 1, v_{4k+2} \leftarrow 1, v_{4k+3} \leftarrow -1$

**Case (iii): when** $m \equiv 2 \pmod{4}$

1. $v_{m+1} \leftarrow -1, v'_{m+1} \leftarrow 1$

2. For $1 \leq k \leq \frac{m+2}{4}$:
   - $v_{4k-2} \leftarrow 1, v_{4k-1} \leftarrow -1, v_{4k} \leftarrow -1$
   - $v'_{4k-3} \leftarrow 1, v'_{4k-2} \leftarrow 1, v'_{4k-1} \leftarrow 1, v'_{4k} \leftarrow 1$

3. For $\frac{m+2}{4} \leq k \leq \frac{m-2}{4}$:
   - $v_{4k} \leftarrow 1, v_{4k+1} \leftarrow 1, v_{4k+2} \leftarrow -1, v_{4k+3} \leftarrow -1$

4. For $\frac{m-2}{4} \leq k \leq \frac{m-1}{2}$:
   - $v_{4k} \leftarrow -1, v_{4k+1} \leftarrow 1, v_{4k+2} \leftarrow 1, v_{4k+3} \leftarrow -1$

**Case (iv): when** $m \equiv 3 \pmod{4}$

1. $v_{4k-2} \leftarrow 1, v_{4k-1} \leftarrow -1, v_{4k} \leftarrow -1$

2. For $1 \leq k \leq \frac{m-1}{2}$:
   - $v_{4k-3} \leftarrow 1, v_{4k-2} \leftarrow 1, v_{4k-1} \leftarrow -1, v_{4k} \leftarrow -1$
   - $v'_{4k-3} \leftarrow -1, v'_{4k-2} \leftarrow 1, v'_{4k-1} \leftarrow 1, v'_{4k} \leftarrow 1$

3. For $k = \frac{m+1}{2}$:
   - $v_{4k} \leftarrow 1, v_{4k+1} \leftarrow 1, v_{4k+2} \leftarrow 1, v_{4k+3} \leftarrow -1$

4. For $\frac{m-1}{2} \leq k \leq \frac{m-1}{2}$:
   - $v_{4k} \leftarrow -1, v_{4k+1} \leftarrow 1, v_{4k+2} \leftarrow 1, v_{4k+3} \leftarrow -1$
Theorem 3.3: The extended duplicate graph of double star $EDG(DS_{m,m})$, $m \geq 2$, admits signed product cordial labeling.

Proof:

Let $V = \{v_1, v_2, \ldots, v_{2m+1}, v'_1, v'_2, \ldots, v'_{2m+1}\}$ and $E = \{e_1, e_2, \ldots, e_{2m+1}, e'_1, e'_2, \ldots, e'_{2m}\}$ be the set of vertices and the set of edges of $EDG(DS_{m,m})$.

**Case(i): when $m \equiv 0 \pmod{4}$**

Using the algorithm 3.6, each of the $2m + 1$ vertices receive label $-1$ and $1$. Using the induced function $f^*$ as in theorem 3.1, the $2m$ edges namely $e_2, e_3, e_6, e_7, \ldots, e_{m-6}, e_{m-5}, e_{m-4}, e_{m-3}, e_{m-2}, e_{m-1}, e_{m}, e_{m+1}, e_{m+2}, e_{m+3}$, $e_{m+4}, e_{m+5}, e_{m+6}, e_{m+7}, \ldots$, receive label $1$ and the $2m + 1$ edges namely, $e_1, e_4, e_5, \ldots, e_{m-3}, e_{m-2}, e_{m-1}, e_{m}, e_{m+1}, e_{m+2}, e_{m+3}, \ldots$, receive label $-1$ respectively. Thus the number of edges labeled with $-1$ and $1$ differ by one.

**Case(ii): when $m \equiv 1 \pmod{4}$**

Using the algorithm 3.6, each of the $2m + 1$ vertices receive label $-1$ and $1$. Using the induced function $f^*$ as in theorem 3.1, the $2m$ edges namely $e_2, e_3, e_6, e_7, \ldots, e_{m-6}, e_{m-5}, e_{m-4}, e_{m-3}, e_{m-2}, e_{m-1}, e_{m}, e_{m+1}, e_{m+2}, e_{m+3}$, $e_{m+4}, e_{m+5}, e_{m+6}, e_{m+7}, \ldots$, receive label $1$ and the $2m + 1$ edges namely $e_1, e_4, e_5, e_6, e_9, \ldots, e_{m-1}, e_{m}, e_{m+1}, e_{m+2}, e_{m+3}, \ldots$, receive label $-1$ respectively. Thus the number of edges labeled with $-1$ and $1$ differ by one.

**Case(iii): when $m \equiv 2 \pmod{4}$**

Using the algorithm 3.6, each of the $2m + 1$ vertices receive label $-1$ and $1$. Using the induced function $f^*$ as defined in theorem 3.1, the $2m$ edges namely $e_2, e_3, e_6, e_7, \ldots, e_{m-6}, e_{m-5}, e_{m-4}, e_{m-3}, e_{m-2}, e_{m-1}, e_{m}, e_{m+1}, e_{m+2}, e_{m+3}$, $e_{m+4}, e_{m+5}, e_{m+6}, e_{m+7}, \ldots$, receive label $1$ and the $2m + 1$ edges namely $e_1, e_4, e_5, e_6, e_9, \ldots, e_{m-1}, e_{m}, e_{m+1}, e_{m+2}, e_{m+3}, \ldots$, receive label $-1$ respectively. Thus the number of edges labeled with $-1$ and $1$ differ by one.

**Case(iv): when $m \equiv 3 \pmod{4}$**

Using the algorithm 3.6, each of the $2m + 1$ vertices receive label $-1$ and $1$. Using the induced function $f^*$ as in theorem 3.1, the $2m$ edges namely, $e_2, e_3, e_6, e_7, \ldots, e_{m-1}, e_{m}, e_{m+1}, e_{m+2}, e_{m+3}, \ldots$, receive label $1$ and the $2m + 1$ edges namely $e_1, e_4, e_5, e_6, e_9, \ldots, e_{m-2}, e_{m-1}, e_{m+1}, e_{m+2}, e_{m+3}, \ldots$, receive label $-1$. Thus the number of edges labeled with $-1$ and $1$ differ by one.

Hence, the extended duplicate graph of double star $EDG(DS_{m,m})$, $m \geq 2$, admits signed product cordial labeling.

Illustration:

![Signed product cordial labeling in EDG(DS_{m,m})](image_url)
IV. Conclusion:

We proved that the duplicate graph of triangular ladder, the extended duplicate graph of bistar and the extended duplicate graph of the double star admit signed product cordial labeling.

V. References: