Fractional derivative formulae involving the generalized Lauricella function, the
generalized polynomials and the multivariable Aleph-function II

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ABSTRACT

In this document, we derive two general fractional derivative formulae involving the
generalized Lauricella function, the general polynomials and the
multivariable Aleph-function have been derivative by using the concept of fractional derivatives in the theory of hypergeometric function

KEYWORDS: Aleph-function of several variables, generalized Lauricella function, contour integral, general polynomial, fractional derivative

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1. Introduction and preliminaries.

The function multivariable Aleph-function generalize the multivariable I-function recently study by C.K. Sharma and
Ahmad [6], itself is a generalisation of G and H-functions of multiple variables. The multiple Mellin-Barnes integral occurring in this paper will be referred to as the multivariables Aleph-function throughout our present study and will be defined and represented as follows.

We have: $\mathbb{R}(z_1, \cdots, z_r) = \mathbb{R}_{p_1, q_1, r_1}(z_1; \cdots; \cdots; z_r)$

\[
\left[ (a_j^{(1)}, \cdots, a_j^{(r)})_{1, n}, (\tau_i^{(1)}, \cdots, \tau_i^{(r)})_{n+1, p_i} \right]: \left[ (\beta_j^{(1)}, \cdots, \beta_j^{(r)})_{m+1, q_i} \right]
\]

\[
\left[ (c_j^{(1)}, \gamma_j^{(1)})_{1, m+1}, (\tau_i^{(1)}, (d_j^{(1)}, \delta_j^{(1)})_{m+1, q_i}) \right]: \left[ (\delta_j^{(r)}, \gamma_j^{(r)})_{1, n+1, p_i} \right]
\]

\[
\psi(s_1, \cdots, s_r) = \frac{1}{(2\pi \omega)^r} \int_{L_1} \cdots \int_{L_r} \prod_{k=1}^r \phi_k(s_k) z_k^{s_k} \, ds_1 \cdots ds_r
\]

with $\omega = \sqrt{-1}$

\[
\psi(s_1, \cdots, s_r) = \prod_{j=1}^r \prod_{i=n+1}^{p_i} \Gamma(1 - a_j + \sum_{k=1}^r \alpha_j^{(k)} s_k) \prod_{j=1}^{q_i} \Gamma(1 - b_j + \sum_{k=1}^r \beta_j^{(k)} s_k)
\]

and $\phi_k(s_k) = \prod_{j=1}^{m_k} \Gamma(\delta_j^{(k)} s_k) \prod_{j=1}^{n_k} \Gamma(1 - \alpha_j^{(k)} s_k)$

\[
\sum_{i=1}^{R(k)} \prod_{j=m_k+1}^{q_i^{(k)}} \Gamma(1 - d_j^{(k)} + \gamma_j^{(k)} s_k) \prod_{j=n_k+1}^{p_i} \Gamma(1 - c_j^{(k)} + \gamma_j^{(k)} s_k)
\]

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Suppose, as usual, that the parameters

\[ \alpha_j, j = 1, \cdots, p; \beta_j, j = 1, \cdots, q; \]

\[ c_j^{(k)}, j = 1, \cdots, n_k; c_j^{(k)}, j = n_k + 1, \cdots, p_i^{(k)}; \]

\[ d_j^{(k)}, j = 1, \cdots, m_k; d_j^{(k)}, j = m_k + 1, \cdots, q_i^{(k)}; \]

with \( k = 1, \cdots, r, i = 1, \cdots, R, i^{(k)} = 1, \cdots, R^{(k)} \)

are complex numbers, and the \( \alpha' s, \beta' s, \gamma' s \) and \( \delta' s \) are assumed to be positive real numbers for standardization purpose such that

\[
U_i^{(k)} = \sum_{j=1}^{n} \alpha_j^{(k)} + \tau_i \sum_{j=n+1}^{p_i} \alpha_j^{(k)} + \sum_{j=1}^{m_k} \gamma_j^{(k)} + \sum_{j=n_k+1}^{p_i^{(k)}} \gamma_j^{(k)} - \tau_i \sum_{j=1}^{q_i^{(k)}} \beta_j^{(k)} - \sum_{j=1}^{m_k} \delta_j^{(k)} - \tau_i \sum_{j=m_k+1}^{q_i^{(k)}} \delta_j^{(k)} \leq 0 \tag{1.4}
\]

The numbers \( \tau_i \) are positives for \( i = 1 \) to \( R \), \( \tau_i^{(k)} \) are positives for \( i^{(k)} = 1 \) to \( R^{(k)} \)

The contour \( L_k \) is in the \( s_k \)-plane and run from \( \sigma - \infty \) to \( \sigma + \infty \) where \( \sigma \) is a real number with loop, if necessary, ensure that the poles of \( \Gamma(c_j^{(k)} - \delta_j^{(k)} s_k) \) with \( j = 1 \) to \( m_k \) are separated from those of \( \Gamma(1 - \alpha_j + \sum_{i=1}^{r} c_j^{(k)} s_k) \) with \( j = 1 \) to \( n \) and \( \Gamma(1 - c_j^{(k)} + \gamma_j^{(k)} s_k) \) with \( j = 1 \) to \( n_k \) to the left of the contour \( L_k \). The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as:

\[ |\arg z_k| < \frac{1}{2} \Lambda_i^{(k)} \pi, \text{ where} \]

\[
A_i^{(k)} = \sum_{j=1}^{n} \alpha_j^{(k)} - \tau_i \sum_{j=n+1}^{p_i} \alpha_j^{(k)} - \tau_i \sum_{j=1}^{q_i} \beta_j^{(k)} + \sum_{j=1}^{m_k} \gamma_j^{(k)} - \tau_i \sum_{j=n_k+1}^{p_i^{(k)}} \gamma_j^{(k)} + \sum_{j=1}^{m_k} \delta_j^{(k)} - \tau_i \sum_{j=m_k+1}^{q_i^{(k)}} \delta_j^{(k)} > 0, \text{ with } k = 1, \cdots, r, i = 1, \cdots, R, i^{(k)} = 1, \cdots, R^{(k)} \tag{1.5}
\]

The complex numbers \( z_k \) are not zero. Throughout this document, we assume the existence and absolute convergence conditions of the multivariable Aleph-function.

We may establish the asymptotic expansion in the following convenient form:

\[ \mathfrak{N}(z_1, \cdots, z_r) = 0(|z_1|^{\alpha_1} \cdots |z_r|^{\alpha_r}), \max(|z_1|, \ldots |z_r|) \to 0 \]

\[ \mathfrak{N}(z_1, \cdots, z_r) = 0(|z_1|^{\beta_1} \cdots |z_r|^{\beta_r}), \min(|z_1|, \ldots |z_r|) \to \infty \]

where, with \( k = 1, \cdots, r : \alpha_k = \min[\Re(d_j^{(k)} / \delta_j^{(k)})], j = 1, \cdots, m_k \) and

\[ \beta_k = \max[\Re(c_j^{(k)} - 1 / \gamma_j^{(k)})], j = 1, \cdots, n_k \]
We will use the following notations in this document:

\( V = m_1, n_1; \cdots ; m_r, n_r \)  

\( W = p_i(1), q_i(1), \tau_i(1); R^{(1)}, \cdots , p_i(r), q_i(r), \tau_i(r); R^{(r)} \)  

\( A = \{(a_j; \alpha_j^{(1)}; \cdots ; \alpha_j^{(r)})_{1,n}\}, \{\tau_i(a_{j,i}; \alpha_j^{(1)}; \cdots ; \alpha_j^{(r)})_{n+1, p_i}\} \)  

\( B = \{\tau_i(b_{j,i}; \beta_j^{(1)}; \cdots ; \beta_j^{(r)})_{m+1,q_i}\} \)  

\( C = \{(c_j^{(1)}; \gamma_j^{(1)}),_{1,n}\}, \tau_i(c_j^{(1)}; \gamma_j^{(1)}),_{n+1, p_i}\}, \cdots , \{(c_j^{(r)}; \gamma_j^{(r)}),_{1,n}\}, \tau_i(c_j^{(r)}; \gamma_j^{(r)}),_{n+1, p_i}\} \)  

\( D = \{(d_j^{(1)}; \delta_j^{(1)}),_{1,m}\}, \tau_i(d_j^{(1)}; \delta_j^{(1)}),_{m+1, q_i}\}, \cdots , \{(d_j^{(r)}; \delta_j^{(r)}),_{1,m}\}, \tau_i(d_j^{(r)}; \delta_j^{(r)}),_{m+1, q_i}\} \)

The multivariable Aleph-function write:

\[ \mathcal{N}(z_1, \cdots , z_r) = \mathcal{N}_{p_i,q_i,\tau_i,R;W}^{0,n,V} \left( \begin{array}{c} z_1 \\ \vdots \\ z_r \end{array} \right| \begin{array}{c} A : C \\ \vdots \\ B : D \end{array} \)  

In the present paper, we will use the following results:

Let \( F(\cdots) \) denote the generalized Lauricella function of several complex variables, see Srivastava et al [10]. We have

\[ F \left( \begin{array}{c} x_1 \\ \vdots \\ x_r \end{array} \right) = \sum_{m_1,\cdots ,m_r=0}^{\infty} A(m_1,\cdots ,m_r) \frac{x_1^{m_1} \cdots x_r^{m_r}}{m_1! \cdots m_r!} \]  

where:

\[ A(m_1,\cdots ,m_r) = \frac{\prod_{j=1}^{m_1} (a_j) \theta_j^{r} \cdots m_r \phi_j^{r} \prod_{j=1}^{m_1} (b_j) \phi_j^{m_r} \cdots \prod_{j=1}^{m_1} (c_j) \delta_j^{m_1}}{\prod_{j=1}^{m_1} (d_j) \phi_j^{m_1} \cdots \prod_{j=1}^{m_1} (e_j) \delta_j^{m_1}} \]  

The generalized polynomials defined by Srivastava [8], is given in the following manner:

\[ S^{M_1,\cdots ,M_r}_{N_1,\cdots ,N_r}[y_1,\cdots ,y_r] = \sum_{K_1=0}^{[N_1/M_1]} \cdots \sum_{K_r=0}^{[N_r/M_r]} \frac{(-N_1)_{M_1, K_1} \cdots (-N_r)_{M_r,K_r}}{K_1! \cdots K_r!} A[N_1, K_1; \cdots ; N_r, K_r] y_1^{K_1} \cdots y_r^{K_r} \]  

Where \( M_1, \cdots , M_r \) are arbitrary positive integers and the coefficients \( A[N_1, K_1; \cdots ; N_r, K_r] \) are arbitrary constants, real or complex.

The fractional derivative of a function \( f(x) \) of a complex order \( \mu \) is defined by Oldham et al ([5], 1974, page 49) in the following manner:
For simplicity, the special sense of the fractional derivative operator \( {}_aD_x^\mu \) when \( a = 0 \), will be written \( D_x^\mu \)

Also we have:

\[
D_x^\mu (x^\lambda) = \frac{d^\mu}{dx^\mu} (x^\lambda) = \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda - \mu + 1)} x^{\lambda - \mu}, \quad \text{Re}(\lambda) > -1
\] (1.16)

and the binomial expansion

\[
(x + \mu)^\lambda = \mu^\lambda \sum_{m=0}^{\infty} \binom{\lambda}{m} \left(\frac{x}{\mu}\right)^m, \quad \left|\frac{x}{\mu}\right| < 1
\] (1.17)

In your investigation, we shall use the following result which may be verified from (1.16), binomial theorem and exponential theorem.

\[
D_x^{\lambda - \mu}[x^{\lambda + m - 1} \prod_{j=1}^{r} (1 - a_j x)^{-\alpha_j}] = x^{\mu + m - 1} \frac{\Gamma(\lambda + m)}{\Gamma(\mu + m)} F_{D}^{(r)}[\lambda, \alpha_1, \cdots, \alpha_r; \mu; a_1 x, \cdots, a_r x]
\] (1.18)

Where \( \text{Re}(\lambda) > 0, \max\{|a_1 x|, \cdots, |a_r x|\} < 1 \), and \( F_{D}^{(r)} \) denotes the Lauricella's hypergeometric function of \( r \)-variables which is the generalization of Appell's function \( F_{q}^{(2)} \) of two variables defined by Lauricella [4].

\[
D_x^{\lambda - \mu}[x^{\lambda + m - 1} \exp(a_r x) \prod_{j=1}^{r} (1 - a_j x)^{-\alpha_j}] = x^{\mu + m - 1} \frac{\Gamma(\lambda + m)}{\Gamma(\mu + m)}
\]

\[
\times \phi_D^{(r)}[\lambda, \alpha_1, \cdots, \alpha_{r-1}; \mu; a_1 x, \cdots, a_r x]
\] (1.19)

Where \( \text{Re}(\lambda) > 0, \max\{|a_1 x|, \cdots, |a_{r-1} x|\} < 1 \), and \( \phi_D^{(r)} \) denotes the confluent form of Lauricella's hypergeometric function \( F_{D}^{(r)} \).

2. The fractional derivative formula

In the present paper, we use the following notations.

\[
A_1 = \left( \frac{-N_1}{K_1} \right)^{M_1} \cdots \left( \frac{-N_r}{K_r} \right)^{M_r} A[N_1, K_1; \cdots; N_r, K_r]
\]

\[
A_r = \sum_{i=1}^{r} \lambda_i \zeta; \quad B_r = \sum_{i=1}^{r} \mu_i \zeta \quad ; \quad g(x_i) = x_i (1 - a_i x)^{-\rho_i}, i = 1, \cdots, r
\]

First formula:

\[
D_x^{(\lambda + A_r) - (\mu + B_r)} \left[ x^{\lambda - 1} (x + \zeta)^{\sigma_F} \begin{pmatrix} g(x_1) \\ \cdots \\ g(x_r) \end{pmatrix} \begin{pmatrix} y_1 x^{\sigma_1} (x + \zeta)^{\sigma_1} \\ \cdots \\ y_r x^{\sigma_r} (x + \zeta)^{\sigma_r} \end{pmatrix} \right] = \sum_{m=0}^{\infty} \sum_{k_1=0}^{[N_1/M_1]} \cdots \sum_{k_r=0}^{[N_r/M_r]} A(k_1, \cdots, k_r) A_1 \prod_{i=1}^{r} \frac{x_i}{k_i! l_i! m_i!}
\]
The validity conditions are the following:

\( a) \ Re(\lambda_i) > 0, \ |arg(x/\zeta)| < \pi, i = 1, \ldots , r \)

\( b) \ Re[\lambda + \sum_{i=1}^{r} \lambda_i \min_{1 \leq j \leq m_i} \frac{d_{j}^{(i)}}{\delta_{j}}] > -1, \ Re[\sigma + \sum_{i=1}^{r} \lambda_i \min_{1 \leq j \leq m_i} \frac{d_{j}^{(i)}}{\delta_{j}}] > -1, j = 1, \ldots , m_k \)

\( c) \ |argz_k| < \frac{1}{2} A_{i}^{(k)} \pi, \) where \( A_{i}^{(k)} \) is given in (1.5)

Second formula

\[ D_{x}^{(\lambda+A_{x})-(\mu+B_{x})} \left[ x^{\lambda-1}(x+\zeta)^{\sigma}e^{x_{r}x_{r}} \left[ \begin{array}{c} \frac{g_{x_{1}}}{} \\ \vdots \\ \frac{g_{x_{r-1}}}{} \\ x_{r} \end{array} \right] \right] = x^{\mu-1}z^{\sigma} \sum_{m=0}^{\infty} \sum_{k_{1}=0}^{[N_{1}/M_{1}]} \sum_{K_{1}=0}^{[N_{r}/M_{r}]} A(k_{1}, \ldots , k_{r}) A_{3}^{(r)} \prod_{i=1}^{r} \frac{(x/\zeta)^{m_{i}k_{i}}y_{i}^{k_{i}}}{k_{i}!m_{i}!} \]

\[ \prod_{i=1}^{r} \left[ \rho_{i}k_{i}l_{i}^{k_{i}}(a_{i}x)^{l_{i}}(a_{i}x)^{l_{i}} \right] \left[ \frac{y_{i}}{l_{i}! \l_{i}!} \right] \left[ N_{i}^{0, n+2; V} \rho_{i}+2, q_{i}+2, \tau_{i}; R; W \right] \left[ \begin{array}{c} z_{1}x^{\mu_{1}}(x+\zeta)^{\lambda_{1}} \\ \vdots \\ z_{r}x^{\mu_{r}}(x+\zeta)^{\lambda_{r}} \end{array} \right] \]

The validity conditions are the following:

\( a) \ Re(\lambda_i) > 0, \ |arg(x/\zeta)| < \pi, i = 1, \ldots , r \)

\( b) \ Re[\lambda + \sum_{i=1}^{r} \lambda_i \min_{1 \leq j \leq m_i} \frac{d_{j}^{(i)}}{\delta_{j}}] > -1, \ Re[\sigma + \sum_{i=1}^{r} \lambda_i \min_{1 \leq j \leq m_i} \frac{d_{j}^{(i)}}{\delta_{j}}] > -1, j = 1, \ldots , m_k \)
Proof:
To prove (2.1), we first applying the definition of the generalized Lauricella function, the generalized polynomials and multivariable Aleph-function by its Mellin-barnes contour integral occurring on the left-hand sides, collect the power of \( x \) and \( x + \zeta \) and applying the binomial expansion (1.17). Changing the order of summations, integrations and fractional derivative operators and making use the formula (1.16) and again if apply the definition (1.1), we shall then get the desired formula (2.1).
Similarly (2.2) can be provided by using the definition of exponential serie.

3. Particular cases

If \( \tau_1 = \tau_2 = \cdots = 1 \), the Aleph-function of several variables degenerate in the I-function of several variables defined by Sharma and Ahmad [6]. For more details, see Tiwari [11].

If \( \tau_1 = \tau_2 = \cdots = 1 \) and \( R = R^{(1)} = \cdots = R^{(r)} = 1 \), the multivariable Aleph-function degenerate in multivariable H-function, see Srivastava et al [9]. We obtain

First formula

\[
D_x^{(\lambda + A_r) - (\mu + B_r)} \left[ x^{\lambda - 1} (x + \zeta) \sigma F \left( \begin{array}{c} g (x_1) \\ g (x_r) \end{array} \right) \right]
\]

\[
= x^{\mu - 1} \zeta \sigma \sum_{m=0}^{\infty} \sum_{k_1, \cdots, k_r=0}^{\infty} \sum_{l_1, \cdots, l_r=0}^{\infty} \sum_{K_1=0}^{[N_1/M_1]} \cdots \sum_{K_r=0}^{[N_r/M_r]} A(k_1, \cdots, k_r) A_1 \prod_{i=1}^{r} \frac{x_i^{(x/i)} m y_i^{K_i}}{k_i! l_i! m!}
\]

\[
\prod_{i=1}^{r} (\beta_i k_i) l_i (a_i x)^{l_i} H^{0, n+2: V}_{p+2, q+2: W} \left( \begin{array}{c} z_1 x^{\mu_1} (x + \zeta) \lambda_1 \\ \vdots \\ z_r x^{\mu_r} (x + \zeta) \lambda_r \end{array} \right)
\]

\[
= (1 - \lambda - m - \sum_{i=1}^{r} (\sigma_i K_i + l_i); \lambda_1, \cdots, \lambda_r), (-\sigma - \sum_{i=1}^{r} \sigma_i K_i; \lambda_1, \cdots, \lambda_r), A' : C' \)
\[
= (1 - \mu - m - \sum_{i=1}^{r} (\sigma_i K_i + l_i); \mu_1, \cdots, \mu_r), (\min_{1 \leq j \leq m_i} \delta_j^{(i)}), B' : D'
\]

(3.1)

The validity conditions are the following:

a) \( \text{Re}(\lambda_i) > 0, |\text{arg}(x/\zeta)| < \pi, i = 1, \cdots, r \)

b) \( \text{Re}[\lambda + \sum_{i=1}^{r} \lambda_i \min_{1 \leq j \leq m_i} \delta_j^{(i)}] > -1, \text{Re}[\sigma + \sum_{i=1}^{r} \sigma_i \min_{1 \leq j \leq m_i} \delta_j^{(i)}] > -1, j = 1, \cdots, m_k \)

c) \( |\text{arg}z_i| < \frac{1}{2} A_i \pi \), where \( A_i = \sum_{j=1}^{n} \alpha_j^{(i)} - \sum_{j=n+1}^{p} \alpha_j^{(i)} - \sum_{j=1}^{q} \beta_j^{(i)} + \sum_{j=n+1}^{p} \gamma_j^{(i)} - \sum_{j=1}^{n} \gamma_j^{(i)} \)
The validity conditions are the following:

(a) $\Re(\lambda_i) > 0$, $|\arg(x/\zeta)| < \pi$, $i = 1, \ldots, r$

(b) $\Re[\lambda + \sum_{i=1}^{r} \lambda_i \min_{1 \leq j \leq m_i} \delta_j^{(i)}] > -1$, $\Re[\sigma + \sum_{i=1}^{r} \lambda_i \min_{1 \leq j \leq m_i} \delta_j^{(i)}] > -1$, $j = 1, \ldots, m_k$

(c) $|\arg z_i| < \frac{1}{2} A_i \pi$, where $A_i = \sum_{j=1}^{n} \alpha_j^{(i)} - \sum_{j=n+1}^{p} \alpha_j^{(i)} - \sum_{j=1}^{q} \beta_j^{(i)} + \sum_{j=n+1}^{p} \gamma_j^{(i)} - \sum_{j=n+1}^{p} \gamma_j^{(i)}$

+ $\sum_{j=1}^{m_i} \delta_j^{(i)} > 0$, with $i = 1, \ldots, r$

4. Conclusion

The aleph-function of several variables presented in this paper, is quite basic in nature. Therefore, on specializing the parameters of this function, we may obtain various other special functions such as I-function of several variables defined by Sharma and Ahmad [6], multivariable H-function, see Srivastava et al [9], the Aleph-function of two variables defined by K.sharma [7], the I-function of two variables defined by Goyal and Agrawal [1,2,3], and the h-function of two variables, see Srivastava et al [9].
Reference


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