Complementary Tree Domination in Splitting Graphs of Graphs

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Abstract- Let \( G = (V, E) \) be a simple graph. A dominating set \( D \) is called a complementary tree dominating set if the induced subgraph \( <V–D> \) is a tree. The minimum cardinality of a complementary tree dominating set is called the complementary tree domination number of \( G \) and is denoted by \( \gamma_{ctd}(G) \). For a graph \( G \), let \( V(G) = \{v : v \in V(G)\} \) be a copy of \( V(G) \). The splitting graph \( Sp(G) \) of \( G \) is the graph with the vertex set \( V(G) \cup V'(G) \) and edge set \( \{uv, u'v', uv : uv \in E(G)\} \). In this paper, complementary tree domination number of splitting graphs of graphs are determined.

Keywords- Dominating set, complementary tree dominating set.

I. INTRODUCTION


Given a graph \( G \), let \( V'(G) = \{v' : v \in V(G)\} \) be a copy of \( V(G) \). The splitting graph \( Sp(G) \) of \( G \) is the graph with the vertex set \( V(G) \cup V'(G) \) and edge set \( \{uv, u'v', uv : uv \in E(G)\} \). For each vertex \( v \in V(G) \), there is a corresponding vertex \( v' \in V(\text{Sp}(G)) \) and each edge \( uv \) of a graph \( G \) produces three edges, \( uv, u'v \) and \( uv' \) in \( \text{Sp}(G) \). Therefore \( G \) is an induced subgraph of \( \text{Sp}(G) \).

In this paper, bounds and exact values of complementary tree domination number of splitting graphs of standard graphs are determined. Also relationship between complementary tree domination number of a graph and its splitting graph is established.

Example 1.1. A graph \( G \) and its splitting graph are given in the following figure.

Theorem 1.1. A ctd-set \( D \) of a connected graph \( G = (V, E) \) is also a ctd-set of \( \text{Sp}(G) \) if and only if

(i) \( <D> \) has no isolated vertices

(ii) For each \( v \in D \), \( N(v) \cap (V–D) \neq \emptyset \)

(iii) \( <V–D> \cong K_2 \) and if \( v_1, v_2 \in V–D, N(v_1) \cap D \cap (N(v_2) \cap D) = \emptyset \).

Proof. Let \( D \) be a ctd-set of both \( G \) and \( \text{Sp}(G) \).

(i) Let \( v \in D \) be an isolated vertex in \( <D> \). Then, its duplicate vertex \( v' \) in \( \text{Sp}(G) \) is not adjacent to any of the vertices in \( D \). Hence, \( D \) is not a ctd-set of \( \text{Sp}(G) \). Therefore, \( <D> \) has no isolated vertices.

(ii) Let there exists a vertex \( v \in D \) such that \( N(v) \cap (V–D) = \emptyset \). Then, its duplicate vertex \( v' \) of \( v \) is isolated in \( <V–D> \).

(iii) If \( P_3 \) (a path on 3 vertices) is an induced subgraph of \( <V–D> \), then \( <\text{Sp}(G)–D> \) contains \( C_4 \). Therefore, \( <V–D> \cong K_2 \). Let \( v_1, v_2 \in V–D \) and \( N(v_1, v_2, 2) \cap (N(v_1, v_2, 2) \cap D) = \emptyset \). Then, there exists a vertex \( u \in D \) such that \( uv_1, uv_2 \in E(G) \) and \( <u, v_1, v_2> \cong C_3 \) in \( \text{Sp}(G) \). Hence, \( (N(v_1) \cap D) \cap (N(v_2) \cap D) = \emptyset \).

Conversely, if (i) is true, then \( D \) is dominating set of \( \text{Sp}(G) \). If (ii) holds, then \( <\text{Sp}(G)–D> \) is connected and if (iii) holds, then \( <\text{Sp}(G)–D> \) is acyclic. Therefore, \( <\text{Sp}(G)–D> \) is a tree. Hence, \( D \) is also a ctd-set of \( \text{Sp}(G) \).

Observation 1.1.

(i) For the cycle \( C_n \), \( \gamma_{ctd}(\text{Sp}(C_n)) = n–2 \), where \( n \geq 6 \). For, if \( V(C_n) = \{v_1, v_2, ..., v_n\} \), then

\[ \gamma_{ctd}(\text{Sp}(C_n)) = n–2 \].
\{v_1, v_2, v_3, \ldots, v_{n-1}\} is a minimum ctd-set of Sp(C_n).

(ii) \(\gamma_{ctd}(Sp(C_3)) = 2 = \gamma_{ctd}(Sp(C_4)) \). \(\gamma_{ctd}(Sp(C_3)) = 4\).

(iii) For a wheel \(W_n\), \(\gamma_{ctd}(Sp(W_n)) = n-1, n \geq 7\).

\(W_n = C_{n-1} + K_1\). For \(n \geq 7\), let \(v_1, v_2, \ldots, v_{n-1}\) be the vertices of degree 3 and \(v\) be the vertex of degree \(n-1\) in \(W_n\). Then, \(\{v_1, v_2, v_3, \ldots, v_{n-1}, v, v'\}\) is a minimum \(\gamma_{ctd}\)-set of \(Sp(W_n)\). Hence, \(\gamma_{ctd}(Sp(W_n)) = n-1, n \geq 7\).

(iv) \(\gamma_{ctd}(Sp(K_n)) = n, n \geq 4\). If \(v_1, v_2, \ldots, v_n\) are the vertices of \(K_n\), then, \(\{v_1, v_2, \ldots, v_n\}\) is a \(\gamma_{ctd}\)-set of \(p(K_n)\).

(v) \(\gamma_{ctd}(Sp(K_{m,n})) = 2m\) where \(m \leq n\) and \(m, n \geq 2\).

Let \(A = \{u_1, u_2, \ldots, u_m\}\) and \(B = \{v_1, v_2, \ldots, v_n\}\) be the bipartition of \(V(K_{m,n})\).

Then, \(\{v_1, u_2, \ldots, u_m, u_1', u_2', \ldots, u_m'\}\) is a \(\gamma_{ctd}\)-set of \(Sp(K_{m,n})\).

(vi) \(\gamma_{ctd}(Sp(K_1 + P_n)) = n, n \geq 3\).

Let \(V(K_1 + P_n) = \{v\}\) and \(V(P_n) = \{v_1, v_2, \ldots, v_n\}\).

Then, \(\{v_1, v_2, \ldots, v_n, v', v''\}\) is a \(\gamma_{ctd}\)-set of \(Sp(K_1 + P_n)\).

II. BOUNDS FOR COMPLEMENTARY TREE DOMINATION NUMBER OF SPLITTING GRAPHS OF GRAPHS

Theorem 2.1. For any connected graph \(G\) with \(p \geq 2, 2 \leq \gamma_{ctd}(Sp(G)) \leq 2p-2\).

Proof. \(Sp(G)\) has 2p vertices and radius of \(Sp(G)\) is atleast 2. Hence, \(\gamma_{ctd}(Sp(G)) \geq 2\). Also, there is no vertex of degree 2p-1 in \(Sp(G)\), \(|D| \leq 2p-2\). Therefore, \(2 \leq \gamma_{ctd}(Sp(G)) \leq 2p-2\).

Theorem 2.2. \(\gamma_{ctd}(Sp(G)) = 2\) if and only if \(G \cong C_4, C_5\) or \(K_2\).

Proof. Let \(D\) be a \(\gamma_{ctd}\)-set of \(Sp(G)\) such that \(|D| = 2\).

Let \(D = \{u, v\}\), where \(u, v \in V(Sp(G))\).

Case 1. \(u\) and \(v\) are vertices of \(G\). Then, \(D\) is also a \(\gamma_{ctd}\)-set of \(G\). By Theorem 1.1, it can be seen that \(G \cong C_4, C_5\) or \(K_2\).

Case 2. Let \(u \in V(G)\) and \(v \in V'\).

Subcase 2.1. \(v = u'\).

That is, \(D = \{u, u'\}\) is a \(\gamma_{ctd}\)-set of \(Sp(G)\) and \(G \cong C_3\).

Subcase 2.2. \(v \neq u'\).

Let \(v = v'\), for some \(w \in V(G)\) and \(w' \neq u'\). Then, \(D = \{u, w'\}\) is a \(\gamma_{ctd}\)-set of \(Sp(G)\). \(w' \neq u'\) implies that \(u'\) is not adjacent to both \(u\) and \(w'\), which is not true. Therefore, \(v = u'\).

Case 3. \(u = w'\) and \(v = x'\), where \(w, x \in V(G)\).

If \(p \geq 3\), then any vertex of \(V'(G) - \{w', x'\}\) is not adjacent to both \(w', x'\).

Therefore, \(p = 2\) and hence \(G \cong K_2\).

Conversely, if \(G \cong C_4, C_5\) or \(K_2\), then \(\gamma_{ctd}(Sp(G)) = 2\).

In the following, \(\gamma_{ctd}(Sp(G)) = 2p-2\) is found for the graph \(G\).

Theorem 2.3. \(\gamma_{ctd}(Sp(G)) = 2p-2\) if and only if \(G \cong K_2\).

Proof. Assume, \(\gamma_{ctd}(Sp(G)) = 2p-2\). Let \(D\) be a \(\gamma_{ctd}\)-set of \(Sp(G)\) having 2p-2 vertices.

Let \(V(Sp(G)) - D = \{u, v\}\). Since \(\langle V(Sp(G)) - D \rangle \cong K_2\), and \(v, u \notin V'(G)\), either

(i) \(u, v \in V(G)\) or

(ii) \(u \in V(G)\) and \(v = w'\), for some \(w \in V(G), w \neq u\).

Case 1. \(u, v \in V(G)\).

Let \(w \in D\) then \(w' \in D\). If both \(u\) and \(v\) are adjacent to \(w\), then \(w'\) is adjacent to both \(u\) and \(w\) is adjacent to both \(u'\) and \(v'\). Then, \(D - \{u'\}\) is a ctd-set of \(Sp(G)\), which is a contradiction.

Let exactly one of \(u\) and \(v\) (say \(u\)) is adjacent to \(w\). Then, also \(D - \{u'\}\) is a ctd-set of \(Sp(G)\).

Therefore, no vertex of \(V(G)\) is an element of \(D\) and hence, \(u', v' \in D\). Hence, \(G \cong K_2\).

Case 2. \(u \in V(G)\) and \(v = w'\), for some \(w \in V(G), w \neq u\).

Since \(\langle V(Sp(G)) - D \rangle \cong K_2\), \(w'\) is adjacent to \(u\) in \(\langle V(Sp(G)) - D \rangle\).

That is, \(w\) is adjacent to \(u\) in \(G\). Since \(D\) is a ctd-set, \(w'\) is adjacent to some vertex, say \(x\) in \(D\), \(x \in V(G)\). Then, \(x'\) is adjacent to \(w\) in \(D\). \(w'\) is adjacent to \(x\) implies that \(w\) is adjacent to \(x\) in \(V(G)\) and hence in \(Sp(G)\).

(i) If \(u\) is also adjacent to \(x\), then \(D - \{w\}\) is a ctd-set of \(Sp(G)\).

(ii) If \(u\) is adjacent to some vertex, say \(y\) in \(D\), then \(D - \{x\}\) is a ctd-set of \(Sp(G)\).

(iii) If \(w'\) is adjacent to some vertex, say \(z\) in \(D\), then \(D - \{x\}\) is a ctd-set of \(Sp(G)\).

Therefore, \(u\) is adjacent to \(w\) and \(w'\) is adjacent to \(x\) only. In this case \(G \cong \langle u, w, x \rangle \cong P_3\), a path on three vertices and \(\{u', v', w'\}\) is a \(\gamma_{ctd}\)-set of \(P_3\) and hence, \(\gamma_{ctd}(Sp(G)) = 3 \geq 2p-2\).

Hence, Case 2 is not possible.

From Case 1, it is concluded that \(G \cong K_2\).

Conversely, if \(G \cong K_2\), then \(\gamma_{ctd}(Sp(G)) = 2p-2\).

Remark 2.1. If \(p \geq 3\), then \(\gamma_{ctd}(Sp(G)) \leq 2p-3\).

Equality holds, if \(G \cong P_3\).

Observation 2.1. \(\gamma_{ctd}(G) \leq \gamma_{ctd}(Sp(G))\).

Equality holds, if \(G \cong K_4 - e\).

Theorem 2.4. If \(G\) is a connected graph such that \(\delta(G) \geq 2\), then \(\gamma_{ctd}(Sp(G)) \leq 2p-4\).

Proof. Let \(e = (u, v) \in E(G)\), where \(u, v \in V(G)\).

Let \(D = \{u, v, u', v'\} \subseteq V(Sp(G))\) and let \(D' = V(Sp(G)) - D\).

Since \(\delta(G) \geq 2\), each vertex in \(V(Sp(G)) - D'\) (\(= D\)) is adjacent to at least one vertex in \(D'\) and \(\langle V(Sp(G)) - D' \rangle \cong P_3\) in \(Sp(G)\) and hence, \(D'\) is a ctd-set of \(Sp(G)\).

Therefore, \(\gamma_{ctd}(Sp(G)) \leq |D'| = |V(D)| = 2p-4\).
Equality holds, if $G \cong K_4$, the complete graph on 4 vertices.

**Corollary 2.1.** Let $G$ be a connected graph such that $\delta(G) \geq 2$. If $G$ contains a $P_3$ as an induced subgraph such that central vertex of $P_3$ is of degree at least 3 and other two vertices in $P_3$ is degree at least 2, then $\gamma_{ctd}(Sp(G)) \leq 2p-5$.

**Proof.** Let $V(P_3) = \{u, v, w\}$. Let $D = \{u, v, w, u', w'\}$ and then $D' = V(Sp(G)) - D$ is a $ctd$-set of $Sp(G)$. Therefore, $\gamma_{ctd}(Sp(G)) \leq 2p-5$.

**Theorem 2.5.** Let $G$ be a connected, non complete graph with $\delta(G) \geq 3$, then $\gamma_{ctd}(Sp(G)) \leq 2p-5$.

**Proof.** Since $G$ is not complete, $G$ contains a $P_3$ as an induced subgraph. Let the vertices of $P_3$ be $u, v$ and $w$ where $v$ is the central vertex of $P_3$. Let $D = \{u, v, w, u', w'\}$ and then $D' = V(Sp(G)) - D$. Since $\delta(G) \geq 3$, each vertex in $V(Sp(G)) - D'$ is adjacent to some vertex in $D'$ and $V(Sp(G)) - D' \cong K_{1,4}$ with $v$ as the central vertex of $K_{1,4}$ in $Sp(G)$. Therefore, $D'$ is a $ctd$-set of $Sp(G)$ and hence, $\gamma_{ctd}(Sp(G)) \leq 2p-5$. Equality holds, if $G \cong K_{p+e}$. □

**Theorem 2.6.** Let $G$ be a connected graph such that $\delta(G) \geq 2$, then $\gamma_{ctd}(Sp(G)) \leq 2p-\delta(G)-1$.

**Proof.** Let $v$ be a vertex of maximum degree in $G$. Let $S = \{u \in V(Sp(G)) : u \in N(v)\}$ and $D' = V(Sp(G)) - S - \{v\}$. Then, $V(Sp(G)) - D' = S \cup \{v\}$. Let $u \in N(v)$. Since $\delta(G) \geq 2$, $\deg(u) \geq 2$ and hence, $u$ is adjacent to a vertex of $G$ other than $v$. Let $u$ be adjacent to $w$ such that $w \neq v$. Then, $u' \in S$ is adjacent to $w$, where $w$ is a vertex in $D'$. That is, $u' \in S$ is adjacent to a vertex in $D'$. Also, $v$ is adjacent to at least one vertex in $G$ and hence in $S$. Therefore, $D'$ is a dominating set of $V(Sp(G))$. Moreover, $V(Sp(G)) - D' \cong K_{1,\Delta(G)}$ and hence, $D'$ is a $ctd$-set of $Sp(G)$. Therefore,

$$\gamma_{ctd}(Sp(G)) \leq |V(Sp(G)) - S - \{v\}| = 2p - \Delta(G) - 1.$$  

Equality holds, if $G \cong K_{p+e}$, $p \geq 4$.

**Theorem 2.7.** For any connected graph $G$ with $p$ vertices, $\gamma_{ctd}(Sp(G)) \leq p + \gamma_{ctd}(G)$.

**Proof.** Let $D$ be a minimum $ctd$-set of $G$ and hence, $|D| = \gamma_{ctd}(G)$. Therefore, $<V(G)> - D$ is a tree.

Now, the set $D' = D \cup <V(G)>$ is a $ctd$-set of $Sp(G)$. Hence, $\gamma_{ctd}(Sp(G)) \leq |D'| = \gamma_{ctd}(G) + p$. □

**Theorem 2.8.** If $\gamma_{ctd}(G) = 1$ then $\gamma_{ctd}(Sp(G)) \leq t+1$, where $t$ is the number of vertices of $G$ of degree at least 2.

**Proof.** Assume $\gamma_{ctd}(G) = 1$. Then $G \cong T + K_1$, where $T$ is a tree with at least two vertices.

Let $V(K_1) = \{v\}$ and let $D' = \{v' : \deg(v') \geq 2\}$. Then, $D' \subseteq V(\text{Sp}(G))$ and $|D'| = t-1$, where $t$ is the number of vertices of $G$ of degree at least 2. Let $D = \{v, v'\} \cup D'$, then, $D \subseteq V(\text{Sp}(G))$ and all the vertices in $<V(\text{Sp}(G))> - D$ are adjacent to $v$ and $<V(\text{Sp}(G))> - D$ is the tree obtained from the tree $T$ by attaching $m$ pendant edges at each of the supports $u$ of $T$, where $\deg_G(u) = m, m \geq 1$.

Therefore, $D$ is a $ctd$-set of $Sp(G)$ and hence, $\gamma_{ctd}(Sp(G)) \leq |D| = t+1$. Equality holds, if $G \cong P_n + K_1$ and $G \cong K_{1\times n} + K_1, n \geq 2$.

**Theorem 2.9.** Let $D$ be a minimum $ctd$-set of a connected graph $G$ with least number of edges. Let $S_1, S_2, ..., S_r \geq 1$ be the star decomposition of $V(G)$ such that $|V(S_i)| \geq 2, i = 1, 2, ..., r$. Then, $\gamma_{ctd}(Sp(G)) \leq 2\gamma_{ctd}(G) + r$.

**Proof.** Let $T' = \{u' : u \in D\}$ and $T = \{x_i : x_i$ is the centre of $S_i\}$. For each star $S_i$, with claws $y_1, y_2, ..., y_{u_i}$, the corresponding vertices $y'_1, y'_2, ..., y'_{u_i}$ are dominated by those vertices in $D$ dominating $y_1, y_2, ..., y_{u_i}$ in $<V(G)>$. Then, the set $D' = D \cup T \cup T'$ is a $ctd$-set of $Sp(G)$. Therefore

$$\gamma_{ctd}(Sp(G)) \leq |D'| + |T| + |T'| = \gamma_{ctd}(G) + \gamma_{ctd}(G) + r = 2\gamma_{ctd}(G) + r,$$

That is, $\gamma_{ctd}(Sp(G)) \leq 2\gamma_{ctd}(G) + r$.

The above bound is attained, if $G \cong K_{r+e}$. Let $v_1, v_2, v_3, v_4$ be the vertices of $K_{r+e}$, where $v_1$ and $v_2$ have degree 3 and $v_3$ and $v_4$ have degree 2. The set $D = \{v_i\}$ is a minimum $ctd$-set of $K_{r+e}$, and hence, $\gamma_{ctd}(G) = 1$.

Also, $<V(G)> \cong P_3$, with $v_3$ as the central vertex. Then, $D'=\{v_1, v'_1, v'_2\}$ is a minimum $ctd$-set of $Sp(G)$. Hence, $\gamma_{ctd}(Sp(G)) = 3 \leq 2\gamma_{ctd}(G) + 1$. □

**Theorem 2.10.** Let $G$ be a unicyclic graph. Then, $\gamma_{ctd}(G) = \gamma_{ctd}(Sp(G))$ if and only if $G \cong C_n, n \neq 3, 5$ and $G$ is the graph obtained by attaching pendant edges at exactly one vertex of $C_n$.

**Proof.** Assume $\gamma_{ctd}(G) = \gamma_{ctd}(Sp(G))$.

**Case 1.** The cycle in $G$ is $C_n$.

If $\gamma_{ctd}(G) = 1$, then $\gamma_{ctd}(Sp(G)) = 2$. Hence, $G \cong C_n$.

Let the vertices of $C_n$ be $v_1, v_2, v_3$. Let a pendant edge be attached at exactly one vertex of $C_n$, say at $v_1$. Let the pendant edge be $(v_1, v_4)$. Then, $\gamma_{ctd}(Sp(G)) = 3$ is a minimum $ctd$-set of $Sp(G)$ and hence, $\gamma_{ctd}(Sp(G)) = 4$, whereas $\gamma_{ctd}(G) = 2$. Hence,
\[ \gamma_{ctd}(G) > \gamma_{ctd}(Sp(G)). \] Similarly, if either two or more edges are attached at exactly one vertex of \( C_5 \) (or) pendant edges are attached at vertices of \( C_5 \), then also \( \gamma_{ctd}(Sp(G)) > \gamma_{ctd}(G) \).

**Case 2.** The cycle in \( G \) is \( C_4 \).

If \( G = C_4 \), then \( \gamma_{ctd}(G) = \gamma_{ctd}(Sp(G)) = 2 \). Let the vertices of \( C_4 \) be \( v_1, v_2, v_3, v_4 \) in order. Then, \( \gamma_{ctd}(G) = m + 2 = \gamma_{ctd}(Sp(G)) \).

(i) Let \( m \) \((m \geq 1)\) pendant edges be attached at exactly one vertex of \( C_4 \), say at \( v_i \), then the set consisting of \( v_1, v_4 \) and the pendant vertices forms a minimum ctd-set of \( G \), whereas the set consisting of \( v_1, v_4 \) together with the duplicate vertices corresponding to pendant vertices in \( G \) forms a minimum ctd-set of \( Sp(G) \). Hence, \( \gamma_{ctd}(G) = m + 2 = \gamma_{ctd}(Sp(G)) \).

(ii) If exactly one pendant edge is attached at each of two or more vertices of \( C_4 \), then \( \gamma_{ctd}(Sp(G)) > \gamma_{ctd}(G) \), since if \( G \) contains a path \( P_3 \) as an induced subgraph, \( Sp(G) \) contains \( C_4 \) as an induced subgraph. Therefore, for each \( P_3 \) in \( G \), a vertex is to be added in the ctd-set \( D' \) of \( Sp(G) \), for \( \gamma_{ctd}(Sp(G)) - D' \) to be a tree.

**Case 3.** The cycle in \( G \) is \( C_5 \).

If \( G \equiv C_5 \), then \( \gamma_{ctd}(G) = 3 \) and \( \gamma_{ctd}(Sp(G)) = 4 \). Also, if one or more pendant edges are attached at the vertices of \( C_5 \), then \( \gamma_{ctd}(Sp(G)) > \gamma_{ctd}(G) \).

**Case 4.** \( G \) contains \( C_n \) \((n \geq 6)\) as the unique cycle.

If \( G \equiv C_n \) \((n \geq 6)\) then \( \gamma_{ctd}(G) = \gamma_{ctd}(Sp(G)) = n - 2 \). As in Case 2, if one or more pendant edges are attached at least one of the vertices of \( C_5 \), then \( \gamma_{ctd}(Sp(G)) > \gamma_{ctd}(G) \).

The same result holds, if paths of length atleast 2 are attached at the vertices of \( C_n, n \geq 3 \). From the above cases, it is concluded that, \( \gamma_{ctd}(G) = \gamma_{ctd}(Sp(G)) \) if \( G \equiv C_n \) \((n \neq 3, 5)\) and \( G \) is the graph obtained by attaching pendant edges at exactly one vertex of \( C_4 \).

Converse follows easily. \( \square \)

**Theorem 2.11.** Let \( G \) be a connected graph with \( p \) vertices \((p \geq 3)\) \( V(Sp(G)) = V(G) \cup V'(G) \). Then, \( V'(G) \) is a ctd-set of \( G \) if and only if \( G \) is a tree.

**Proof.** Assume \( V'(G) \) is a ctd-set of \( Sp(G) \). Then, each vertex in \( V(Sp(G)) - V(G) \) is adjacent to atleast one vertex in \( V'(G) \) and \( V(Sp(G)) - V'(G) \) is a tree. That is, \( <V(G)> \) is a tree.

Conversely, assume \( G \) is a tree. Let \( D = V'(G) \). That is, \( D \) contains all the duplicate vertices of \( G \). Since \( G \) is connected, each vertex \( v \) in \( V(Sp(G)) - D = V(G) \) is adjacent to \( \text{deg}_{ctd}(v) \) vertices in \( D \) and \( V(Sp(G)) - D = V(G) \) is a tree. Hence, \( D \) is a ctd-set of \( G \). \( \square \)

**Remark 2.2.** For a tree \( T \) with \( p \) vertices, \( \gamma_{ctd}(Sp(G)) \leq p \). This bound is attained, if \( G \equiv K_{1,n}, n \geq 1 \).

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