Prime Cyclic Rings
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Abstract:
Kleinfeld [1] proves that every prime cyclic ring where 2x=0 implies x=0 is commutative and associative. In this Paper we improve on this result by showing that every prime cyclic ring is associative and commutative without assuming 2x=0 implies x=0.

Key Words: Cyclic Ring, Prime Ring, Semi-prime Ring, Associative Ring and Commutative Ring.

Introduction:
We define a cyclic ring to be a not necessarily associative ring R that satisfies the cyclic identity
\[ x(yz) = y(2x) \]  \hspace{1cm} (1)
For all \( x, y, z \) in R.

A ring is called prime ring if every pair of ideals \( I \) and \( J \) of \( R \), \( IJ=0 \) implies that either \( I=0 \) or \( J=0 \). \( R \) is called semiprime if for every ideal \( I \) of \( R \), \( I^2=0 \) implies \( I=0 \). It is clear that prime implies semiprime.

Main Results:

LEMMA 1: Let \( R \) be a cyclic ring. Then the following identities hold:
(i) \((xy)(st)=(tx)(sy)\),
(ii) \((xy)((st)(uv))=(tx)((sy)(uv))\),
(iii) \((xy, st, uv)=0\),
(iv) \((x, y, st)(uv)=0\).

PROOF: (i) By cyclic identity (1), we have
\[(xy)(st) = s(t(xy)) = s(y(tx)) = (tx)(sy)\] Therefore \((xy)(st) = (tx)(sy)\).

(ii) Using cyclic identity (1) and part (i) we obtain
\[(xy)((st)(uv)) = (uv)((xy)(st)) = (uv)((tx)(sy)) = (tx)((xy)(uv)).\]

(iii) Using cyclic identity (1) use parts (i) and (ii) we obtain
\[(xy)((st)(uv)) = (xy)((vs)(ut)) \quad \text{(from part (i))}\]
\[= (sx)((vy)(st)) \quad \text{(from part (ii))}\]
\[= (vs)((ux)(tv)) \quad \text{(from part (i))}\]
\[= (vs)((yr)(ux)) \quad \text{(from part (i))}\]
\[= (vs)(u(x( yr))) \quad \text{(from 1)}\]
\[= u((x( yr))(vs)) \quad \text{(from 1)}\]
\[= u(v(s(x( yr)))) \quad \text{(from 1)}\]
\[= (s(x( yr)))(uv) \quad \text{(from 1)}\]
\[(yt)(sx)(uv) \text{ (from 1)}\]

\[= ((xy)(st))(uv) \text{ (from part (i))}\]

(iv) We use part (i) and (ii) and (iii) and cyclic identity (1) to obtain

\[((xy)(st))(uv) = (xy)((st)(uv)) \text{ (from part (iii))}\]

\[= (xy)((ys)(ut)) \text{ (from part (i))}\]

\[= (ys)((vy)(ut)) \text{ (from part (ii))}\]

\[= (ys)(xt)(uv) \text{ (from part (i))}\]

\[= ((ys)(xt))(uv) \text{ (from part (iii))}\]

\[=((st)(xy))(uv) \text{ (from part (i))}\]

\[= (x(y(st))(uv) \text{ (from (1))}\]

\[= ((st)(xy))(uv) \text{ (from part (i))}\]

\[= (x(st))(xy) \text{ (from (1))}\]

Therefore \[(x,y,st)(uv)=0.\]

**COROLLARY 1:** Let \(R\) be a cyclic ring. Then \(R^2\) is associative.

**PROOF:** From part (iii) of lemma (1), every element of \(R^2\) is a finite sum of products of elements in \(R\).

**LEMMA 2:** Let \(R\) be a cyclic ring.

(i) If \(R\) satisfies the identity \((x,y,st)=0\), then \(R\) also satisfies the identities \([x,y,st]=0\) and \((x,y,z)(st)=0\).

(ii) If \(R\) is associative, then \(R\) satisfies the identity \([x,y](st)=0\).

**PROOF:** (i) Let us assume that \(R\) satisfies the identity \((x,y,st)=0\).

Then \((xy)(st) = x(y(st)) \text{ (from (2))}\]

\[= (st)(xy) \text{ (from (1))}\]

That is \([x,y,st]=0\). Let \(x,y,z,s,t \in R\) and using (1)

Consider \(((xy)z)(st)\)

\[=((xy)z)(st) = (xy)(z(st))(from (2))\]

By applying (1) to this continuously we have

\[= (z(st))(xy)\]

\[= x(y(z(st)))\]

\[= x((st)(yz))\]

\[= x((yz)(st))\]

\[= (x(yz))(st).\]

Therefore \((x,y,z)(st)=0\).

(ii) Since \(R\) is associative then \((x,y,st)=0\).

By Part (i), if \(R\) satisfies the identity \((x,y,st)=0\) then also satisfies \([xy,xy]=0\).

This implies \((xy)(st)=(st)(xy)\)

Now use part (i) continuously gives

\[= y((st)x)\]

\[= y(s(tx))\]

\[= y(x(st))\]

\[= (yx)(st).\]

Thus \([x,y](st)=0.\)

**LEMMA 3:** Let \(R\) be a cyclic ring and let \(I=\{x \in xR^2 = 0\}\). Then \(I\) is an ideal of \(R\).

**PROOF:** Let \(x \in I, y,s,t \in R\), then by identity (1), we have

\[=(xy)(st) = s(t(xy))\]

\[= s(x(ty))\]

\[= 0.\]

Therefore \(I\) is a right ideal of \(R\).

\[=(yx)(st) = s(t(yx))\]
\[ s(x(ty)) = 0. \]

Therefore \( I \) is a left ideal of \( R \).

Hence \( I \) is a ideal of \( R \).

**THEOREM 1:** Let \( R \) be a prime cyclic ring then \( R \) is associative and commutative.

Proof: Let \( N = I \cap R^2 \). Clearly \( N \) is an ideal of \( R \) (from lemma (3)).

Now \( N^2 \subseteq NR^2 = 0 \).

Since \( R \) is prime, \( N = 0 \).

By lemma (1) part (iv) we have \((x,y,st) \in N \) for all \( x,y,s,t \in R \).

By lemma (2(i)), if \( R \) satisfies \((x,y,st)=0\) then \((x,y,z)(st)=0\).

Which gives \( A.I = 0 \) for all \( s,t \in I \).

Since \( R \) is prime either \( A = 0 \) or \( I = 0 \).

If \( I = (0) \) then \( N = 0 \), implies \( R \) is associative.

If \( A = (0) \) then \( R \) is associative.

**References:**