Properties of nano b-regular spaces

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Abstract:
The purpose of this paper is to introduce a new class of regular spaces namely nano b-regular and nano gb-regular spaces. Some basic properties of these separation axioms are studied by utilizing nano b-open and nano gb-open sets.

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1. INTRODUCTION
By using semi open sets due to Levine[14] Jin Han Park[11] studied the properties of s-normal spaces and some functions. As a generalization of closed sets, in 1970, Levine [13] initiated the study of so called g-closed sets. Using g-closed sets, Munshi [16] introduced g-regular and g-normal spaces in topological spaces. And in 1999, Takashi Noiri[19] investigated the characterizations of \( \alpha \)-regular spaces and the preservation theorems in a topological spaces. Since then many topologists have utilized these concepts to the various notions of subsets, weak separation axioms, weak regularity, weak normality and weaker and stronger forms of covering axioms in the literature. The concept of g-regular spaces in topological spaces was proposed and studied by V. Popa and T. Noiri[20]. Further, Vigilino[22] established the notion of semi-normal spaces and C-compact spaces and obtained its basic properties.

In this paper, we define the two new classes of spaces called nano b-regular spaces and nano gb-regular spaces in nano topological spaces. We bring out several characterizations of these spaces along with already existing weaker forms of regularity and obtain some of the preservation theorems.

2. PRELIMINARIES
Definition 2.1[23]: Let \( U \) be a non-empty finite set of objects called the universe and \( R \) be an equivalence relation on \( U \) named as the indiscernibility relation. Then \( U \) is divided into disjoint equivalence classes. Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair \((U, R)\) is said to be the approximation space.

Let \( X \subseteq U \)
1. The lower approximation of \( X \) with respect to \( R \) is the set of all objects, which can be for certainly classified as \( X \) with respect to \( R \) and is denoted by \( L_R(X) \). That is
   \[ L_R(X) = \bigcup_{x \in U} \{ R(x) : R(x) \subseteq X \} \], where \( R(x) \) denotes the equivalence class determined by \( x \in U \).
2. The upper approximation of \( X \) with respect to \( R \) is the set of all objects, which can be possibly classified as \( X \) with respect to \( R \) and is denoted by \( U_R(X) \). That is
   \[ U_R(X) = \bigcup_{x \in U} \{ R(x) : R(x) \cap X \neq \emptyset \} \]
3. The boundary region of \( X \) with respect to \( R \) is the set of all objects, which can be classified neither as \( X \) nor as not-\( X \) with respect to \( R \) and it is denoted by \( B_R(X) \). That is
   \[ B_R(X) = U_R(X) - L_R(X) \].

Definition 2.2[15]: Let \( U \) be non-empty, finite universe of objects and \( R \) be an equivalence relation on \( U \) named as the indiscernibility relation on \( U \). Let \( X \subseteq U \). Let \( \tau_R(X) = \{ U, \phi, L_R(X), U_R(X), B_R(X) \} \). Then \( \tau_R(X) \) is a topology on \( U \), called as the nano topology with respect to \( X \). Elements of the nano topology are known as the nano-sets. \( \tau_R(X) \) is called the nano topological space. \( [\tau_R(X)]^c \) is called as the dual nano topology of \( \tau_R(X) \). Elements of \( [\tau_R(X)]^c \) are called as nano closed sets.

Definition 2.3[5]: Let \( (U, \tau_R(X)) \) be a nano topological space and \( A \subseteq U \). Then \( A \) is said to be nano b-open if \( A \subseteq Ncl(NintA) \cup Nint(NclA) \).

Definition 2.4[6]: Let \( (U, \tau_R(X)) \) and \( (V, \tau_V(Y)) \) be nano topological spaces. Then a mapping \( f : (U, \tau_R(X)) \rightarrow (V, \tau_V(Y)) \) is said to be nano continuous if \( f^{-1}(B) \) is nano open in \( U \) for every nano-open set \( B \) in \( V \).

Definition 2.7: A space \( X \) is said to be p-normal [21] (resp.s-normal [11]) if for any pair of disjoint closed
sets $A$ and $B$, there exist disjoint preopen (resp. semi open) sets $U$ and $V$ such that $A \subseteq U$ and $B \subseteq V$.

**Definition 2.8[7]**: A nano topological space $(U, \tau_R(X))$ is said to be nano b-normal if for any pair of disjoint nano closed sets $A$ and $B$, there exist disjoint nano b-open sets $M$ and $N$ such that $A \subseteq M$ and $B \subseteq N$.

### 3. NANO b-REGULAR SPACES

**Definition 3.1**: A nano topological space $U$ is said to be nano b-regular if for each nano closed set $F$ and a point $u \notin F$, there exist disjoint nano b-open sets $M$ and $N$ such that $u \in M$ and $F \subseteq N$.

**Theorem 3.2**: Every nano regular space is nano b-regular.

**Remark 3.3**: The reverse implication of the above Theorem need not be true can be seen from the following example.

**Example 3.4**: Let $U = \{a, b, c, d\}$ with $U/R = \{\{a\}, \{c\}, \{b, d\}\}$ and $X = \{a, b\}$. Then the nano topology $\tau_R(X) = \{U, \phi, \{a\}, \{b, d\}, \{a, b, d\}\}$ is nano b-regular but not nano regular space.

**Definition 3.5**: A subset $A$ of a nano topological space $(U, \tau_R(X))$ is called

(a) nano generalized b-closed [5](briefly, nano gb-closed), if $\text{Nbc}(A \subseteq G$ whenever $A \subseteq G$

(b) nano gb-open if the complement of $A$ is nano gb-closed in $(U, \tau_R(X))$.

and a subset $A$ of a nano topological space $(U, \tau_R(X))$ is nano gb-open if and only if $G \subseteq \text{Nbc}(A)$ whenever $G \subseteq A$ and $G$ is nano closed in $(U, \tau_R(X))$.

**Definition 3.6**: A function $f : (U, \tau_R(X)) \rightarrow (V, \tau_R(Y))$ is said to be

(a) almost nano b-open if $f(M)$ is nano b-open in $V$ for every nano regular open set $M$ of $U$

(b) almost nano gb-closed if $f(F)$ is nano gb-closed in $V$ for every nano regular closed set $F$ of $U$

and

(c) nano pre b-closed if $f(F)$ is nano b-closed for every nano b-closed set $F$ of $U$.

**Theorem 3.7**: In a nano topological space, the following conditions are equivalent:

(a) $U$ is nano b-regular.

(b) For every point $u \in U$ and every nano open set $N$ containing $u$ there exists a nano b-open set $M$ such that $u \in M \subseteq \text{NbCl}(M) \subseteq N$.

(c) For every nano closed set $A$, the intersection of all the nano b-closed $b$-neighbourhoods of $A$ is $A$.

(d) For every set $A$ and a nano open set $B$ such that $A \cap B \neq \phi$, there exists a nano b-open set $O$ such that $A \cap O \neq \phi$ and $\text{NbCl}(O) \subseteq B$.

(e) For every non-empty set $A$ and nano closed set $B$ such that $A \cap B = \phi$, there exists disjoint nano b-open sets $P$ and $Q$ such that $A \cap P \neq \phi$ and $B \subseteq Q$.

**Proof:** (a) $\Rightarrow$ (b): Let $N$ be a nano open set containing $u$. Then $U-N$ is nano closed and $u \notin U-N$. Since $U$ is nano b-regular there exist nano b-open sets $L$ and $M$ such that $U - N \subseteq L$, $u \in M$ and $L \cap M = \phi$. Now $\text{NbCl}(U - L) = U-L$, for $M \subseteq U$ and $U-L$ is nano b-closed. Hence, $\text{NbCl}(M) \subseteq N$.

(b) $\Rightarrow$ (c): Let $A$ be nano closed and $u \notin A$. Then, $U-A$ is nano open and contains $u$. By (b) there is a nano b-open set $M$ such that $u \in M \subseteq \text{NbCl}(M) \subseteq U$. And so, $U-M \supseteq U - \text{NbCl}(M) \supseteq A$. Consequently, $U-M$ is nano b-closed nano b-neighbourhood of $A$ to which $u$ does not belong. Hence (c) holds.

(c) $\Rightarrow$ (d): Let $A \cap B \neq \phi$ and $B$ is nano open. Let $u \in A \cap B$. Since $u$ does not belong to the nano closed set $U-B$, there exists a nano b-closed nano b-neighbourhood of $U-B$, say $N$ such that $u \notin N$. Let $U-B \subseteq M \subseteq N$, where $M$ is nano b-open. Then $O = U-N$ is nano b-open set which contains $u$ and so $A \cap O \neq \phi$. Also, $U-M$ being nano b-closed, $\text{NbCl}(O) = \text{NbCl}(U-N) \subseteq U-M \subseteq B$.

(d) $\Rightarrow$ (e): If $A \cap B = \phi$, where $A$ is non-empty and $B$ is nano closed then $A \cap (U-B) \neq \phi$ and $U-B$ is nano open. Therefore by (d) there exists a nano b-open set $P$ such that $A \cap P \neq \phi$, $P \subseteq \text{NbCl}(P) \subseteq U-B$. Put $Q = U - \text{NbCl}(P)$. Then $B \subseteq Q$, $P$ and $Q$ are nano b-open sets, such that $Q = U - \text{NbCl}(P) \subseteq U - P$.

(e) $\Rightarrow$ (a): Obvious.

**Theorem 3.8**: If $f: U \rightarrow V$ is a nano continuous, nano b-open and nano gb-closed surjection and if $U$ is a nano regular space then $V$ is nano b-regular.

**Proof**: Let $v \in V$ and $N$ be a nano open set containing $v$ of $(V, \tau_R(Y))$. Let $u$ be a point of $(U, \tau_R(X))$ such that $v = f(u)$. Since $(U, \tau_R(X))$ is nano regular and $f$ is nano continuous, there is a nano open...
set $M$ such that $u \in M \subset \text{Ncl}(M) \subset f^{-1}(N)$. Hence, $v \in f(M) \subset f(\text{Ncl}(M)) \subset N$. Since $f$ is nano gb-closed map, $f(\text{Ncl}(M))$ is a nano gb-closed set contained in the nano open set $N$. Hence we have, $\tau^*_R - \text{Ncl}(f(\text{Ncl}(M))) \subset N$. Therefore, $v \in f(M) \subset f(\text{Ncl}(f(\text{Ncl}(M)))) \subset N$. This implies $v \in f(M) \subset f(\text{Ncl}(f(M))) \subset N$ and $f(\tau^*_R - \text{Ncl}(M))$ is a nano gb-closed set contained in the nano open set $N$. Hence we have, $\tau^*_R - \text{Ncl}(f(\tau^*_R - \text{Ncl}(M))) \subset N$. Therefore, $v \in f(M) \subset \tau^*_R - \text{Ncl}(f(M)) \subset N$. This implies $v \in f(M) \subset \tau^*_R - \text{Ncl}(f(M)) \subset N$ and $f(M)$ is nano b-open. Hence by Theorem 3.7, $(V, \tau^*_R (Y))$ is nano b-regular.

4. NANO gb-REGULAR SPACES

Definition 4.1: A nano topological space $(U, \tau_R (X))$ is said to be nano gb-regular if for each nano gb-closed set $F$ and a point $u \notin F$, there exists disjoint nano b-open sets $M$ and $N$ such that $u \in M$ and $F \subset N$.

Theorem 4.2: A nano topological space $(U, \tau_R (X))$, the following conditions are equivalent:

(a) $(U, \tau_R (X))$ is nano gb-regular.

(b) Every nano gb-closed set $M$ is a union of nano b-regular sets.

(c) Every nano gb-closed set $A$ is an intersection of nano b-regular sets.

Proof: (a) $\Rightarrow$ (b): Let $M$ be a nano gb-open set and let $u \in M$. If $A= U- M$, then $A$ is nano gb-closed. By assumption there exist disjoint nano b-open subsets $W_1$ and $W_2$ of $U$ such that $u \in W_1$ and $A \subset W_2$. If $N = \text{Ncl}(W_1)$, then $N$ is nano b-closed and $N \cap A \subset N$.

(b) $\Rightarrow$ (c): This is obvious.

(c) $\Rightarrow$ (a): Let $A$ be nano gb-closed and let $u \notin A$. By the hypothesis there exist a nano b-regular set $N$ such that $A \subset N$ and $u \notin N$. If $M = U \setminus N$, then $M$ is nano b-open set containing $u$ and $M \cap N = \emptyset$. Thus, $(U, \tau_R (X))$ is nano gb-regular.

Definition 4.3: A subset $N$ of $U$ is said to be a nano gb-neighbourhood of a point $u$ in $U$, if there exists a nano gb-open set $M$ such that $u \in M \subset N$.

Theorem 4.4: If $B \subseteq A \subseteq U$, $B$ is nano gb-closed relative to $A$ and that $A$ is nano open and nano gb-closed in $(U, \tau_R (X))$, then $B$ is nano gb-closed in $(U, \tau_R (X))$.

Proof: Let $B \subseteq A$ and $A$ is both nano gb-closed and nano open set, then $\text{Ncl}(A) \subseteq A$ and hence $\text{Ncl}(B) \subseteq A$. Now from the fact that $A \cap \text{Ncl}(B) = \text{Ncl}_A(B)$ we have, $\text{Ncl}(B) = \text{Ncl}_A(B) \subseteq A$. If $B$ is nano gb-closed relative to $A$ and $G$ is nano open set of $U$ such that $B \subset G$, then $B = B \cap A \subseteq G \cap A$, where $G \cap A$ is nano open in $A$. As $B$ is nano gb-closed relative to $A$, $\text{Ncl}(B) = \text{Ncl}_A(B) \subseteq G \cap A \subseteq G$. Therefore, $B$ is nano gb-closed in $U$.

Theorem 4.5: If $B \subseteq A \subseteq U$, $B$ is nano gb-closed in $(U, \tau_R (X))$ and that $A$ is nano open and nano gb-closed in $(U, \tau_R (X))$, then $B$ is nano gb-closed relative to $A$.

Proof: Conversely, if $B$ is nano gb-closed in $U$ and $G$ is an nano open set of $A$ such that $B \subset G$, then $G = N \cap A$ for some nano open subset $N$ of $U$. As $B \subseteq N$ and $B$ is nano gb-closed in $U$, $\text{Ncl}(B) \subseteq N$. Thus, $\text{Ncl}_A(B) = \text{Ncl}(B) \cap A \subset N \cap A = G$. Therefore, $B$ is nano gb-closed relative to $A$.

Theorem 4.6: If $(U, \tau_R (X))$ is a nano gb-regular space and $V$ is a nano open and nano gb-closed subset of $(U, \tau_R (X))$, then the space $V$ is nano gb-regular.

Proof: Let $F$ be any nano gb-closed set of $V$ and $v \in F$. By the Theorem 4.4 $F$ is nano gb-closed in $(U, \tau_R (X))$. Since $(U, \tau_R (X))$ is nano gb-regular, there exist disjoint nano b-open sets $M$ and $N$ of $(U, \tau_R (X))$ such that $v \in M$, and $F \subseteq N$. Since $V$ is nano open and hence nano b-open we get $M \cap V$.
and \( N \cap V \) are disjoint nano \( b \)-open sets of the subspace \( V \) such that \( v \in M \cap V \) and \( F \subseteq N \cap V \). Hence the subspace \( V \) is nano gb-regular.

**Theorem 4.7:** Let \((U, \tau_R(X))\) be a nano topological space. Then the following statements are equivalent:

(a) \((U, \tau_R(X))\) is nano gb-regular.

(b) For each point \( u \in U \) and for each nano gb-open neighbourhood \( W \) of \( u \), there exists a nano \( b \)-open set \( M \) of \( U \) such that \( \text{NbCl}(M) \subseteq W \).

(c) For each point \( u \in U \) and for each nano gb-closed set \( F \) not containing \( u \), there exists a nano \( b \)-open set \( N \) of \( U \) such that \( \text{NbCl}(N) \cap F = \emptyset \).

**Proof:**

(a) \( \Rightarrow \) (b): Let \( W \) be any nano gb-open neighbourhood of \( u \). Then there exists a nano gb-open set \( G \) such that \( u \in G \subseteq W \). Since \( G^c \) is nano gb-closed and \( u \notin G^c \), by assumption, there exist nano \( b \)-open sets \( M \) and \( N \) such that \( G^c \subseteq N \), \( u \in M \) and \( M \cap N = \emptyset \). Thus, \( \text{NbCl}(M) \subseteq W \).

(b) \( \Rightarrow \) (c): Let \( F \) be any nano gb-open set and \( u \notin F \). Then \( u \in F^c \) and \( F^c \) is nano gb-open and so \( F^c \) is a nano gb-neighbourhood of \( u \). By assumption, there exists a nano \( b \)-open set \( N \) of \( u \) such that \( u \in N \) and \( \text{NbCl}(N) \subseteq N^c \). Then \( (\text{NbCl}(N))^c \) is nano gb-closed containing \( F \) and \( N \cap (\text{NbCl}(N))^c = \emptyset \). Therefore \( U \) is nano gb-regular.

(c) \( \Rightarrow \) (b): Let \( u \in U \) and \( W \) be a nano gb-neighbourhood of \( u \). Then there exists a nano gb-open set \( G \) such that \( u \in G \subseteq W \). Since \( G^c \) is a nano gb-closed set and \( u \notin G^c \), by assumption there exists a nano \( b \)-open set \( M \) of \( U \) such that \( \text{NbCl}(M) \cap G^c = \emptyset \). Therefore \( \text{NbCl}(M) \subseteq G \subseteq W \).

**Theorem 4.8:** If \((U, \tau_R(X))\) is nano gb-regular space and \( f : (U, \tau_R(X)) \rightarrow (V, \tau_R(Y)) \) is bijective, nano gb- irresolute and nano \( M_b \)-open, then \((V, \tau_R(Y))\) is nano gb-regular.

**Proof:** Let \( v \in V \) and \( F \) be any nano gb-closed subset of \((V, \tau_R(Y))\) with \( v \notin F \). Since \( f \) is nano gb- irresolute, \( f^{-1}(F) \) is nano gb-closed in \((U, \tau_R(X))\). Since \( f \) is bijective, let \( f(u) = v \), then \( u \neq f^{-1}(v) \). By assumption, there exist nano \( b \)-open sets \( M \) and \( N \) such that \( u \in M \) and \( f^{-1}(N) \subseteq N \). Since \( f \) is nano \( M_b \)-open and bijective we have, \( v \in f(M) \) and \( F \subseteq f(N) \) and \( f(M) \cap f(N) = f(M \cap N) = \emptyset \). Hence \((V, \tau_R(Y))\) is nano gb-regular space.

**Definition 4.9:** A function \( f : (U, \tau_R(X)) \rightarrow (V, \tau_R(Y)) \) is called nano gc- irresolute if \( f^{-1}(F) \) is nano g-closed in \( U \) for every nano g-closed set \( F \) in \( V \).

**Theorem 4.10:** If \( f : (U, \tau_R(X)) \rightarrow (V, \tau_R(Y)) \) is nano gc- irresolute, nano \( M_b \)-closed and \( A \) is a nano gb-closed subset of \((U, \tau_R(X))\), then \( f(A) \) is nano gb-closed.

**Theorem 4.11:** If \( f : (U, \tau_R(X)) \rightarrow (V, \tau_R(Y)) \) is nano gc- irresolute, nano \( M_b \)-closed and injective and \((V, \tau_R(Y))\) is nano gb-regular, then \((U, \tau_R(X))\) is nano gb-regular.

**Proof:** Let \( F \) be any nano gb-closed set of \((U, \tau_R(X))\) and \( u \notin F \). Since \( f \) is nano gc- irresolute, nano \( M_b \)-closed then by Theorem 4.10, \( f(F) \) is nano gb-closed in \( V \) and \( f(u) \notin f(F) \). Since \((V, \tau_R(Y))\) is nano gb-regular, and so there exist disjoint nano \( b \)-open sets \( M \) and \( N \) in \((V, \tau_R(Y))\) such that \( f(u) \in M \) and \( f(F) \subseteq N \). By assumption, \( f^{-1}(M) \) and \( f^{-1}(N) \) are nano \( M_b \)-open, such that \( u \in f^{-1}(M) \) and \( F \subseteq f^{-1}(N) \) and \( f^{-1}(M) \cap f^{-1}(N) = \emptyset \). Therefore, \((U, \tau_R(X))\) is nano gb-regular.

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