On *(gr)-Closed Sets in Topological Spaces

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Abstract: In this paper we introduced and study the notions of *(gr)-closed sets and *(gr)-open sets in topological spaces and also we discussed their properties.

Keywords: *(gr)-closed set, *(gr)-open set, ġ-open.

1. Introduction & Preliminaries


Let \((X, \tau)\) be a topological space with no separation axioms are assumed. If \(A \subseteq X\), \(\text{cl}(A)\) and \(\text{int}(A)\) will respectively denote the closure and interior of \(A\) in \((X, \tau)\).

Definition 1.1 A subset \(A\) of a topological space \((X, \tau)\) is called
1) Pre- closed set [9], if \(\text{cl}(\text{int}(A)) \subseteq A\).
2) Semi- closed set [7], if \(\text{int} (\text{cl}(A)) \subseteq A\).
3) Semi- pre closed set [1], if \(\text{int}(\text{cl}(\text{int}(A))) \subseteq A\).
4) Regular closed set [14], if \(A = \text{cl}(\text{int}(A))\).
5) \(\alpha\)-closed set [12], if \(\text{cl}(\text{int}(\text{cl}(A))) \subseteq A\).

Definition 1.2 [13] For any subset \(A\) of \((X, \tau)\), \(\text{rcl}(A) = \cap \{B: B \supseteq A, B\text{ is a regular closed subsets of }X\}\)

Definition 1.3 A subset \(A\) of a topological space \((X, \tau)\) is called
1) g- closed set [8], if \(\text{cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is open in \((X, \tau)\).
2) gs- closed set [2], if \(\text{ scl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is open in \((X, \tau)\).
3) sg- closed set [3], if \(\text{ scl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is semi open in \((X, \tau)\).
4) gα - closed set [10], if \(\text{ acl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is \(\alpha\)-open in \((X, \tau)\).
5) ag- closed set [10], if \(\text{ acl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is open in \((X, \tau)\).
6) gp- closed set [11], if \(\text{ pcl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is open in \((X, \tau)\).
7) gsp- closed set [5], if \(\text{ spcl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is open in \((X, \tau)\).
8) rg- closed set [13], if \(\text{ cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is regular open in \((X, \tau)\).
9) gp- closed set [6], if \(\text{ pcl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is regular open in \((X, \tau)\).
10) \(\tilde{g}\)- closed set [15], if \(\text{ cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is semiopen in \((X, \tau)\).

2. *(gr) - CLOSED SETS

Definition 2.1. A subset \(A\) of a topological space \((X, \tau)\) is called a star generalized regular closed set [briefly *(gr)-closed set ], if \(\text{rcl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is \(\tilde{g}\)-open subset of \(X\).

Theorem 2.2 If \(A\) is \(r\)-closed set in \((X, \tau)\). Then \(A\) is *(gr)- closed set in \((X, \tau)\).

Proof: Let \(A\) be an \(r\)-closed set in \((X, \tau)\). Let \(U\) be a \(\tilde{g}\)-open set such that \(A \subseteq U\). Therefore, we have \(\text{rcl}(A) = A \subseteq U\), implies \(\text{rcl}(A) \subseteq U\). Hence \(A\) is *(gr)-closed in \((X, \tau)\).

The converse of the above theorem need not be true as seen from the following example.
Example 2.3 Let $X = \{a, b, c, d\}$ with $\tau = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\}$. The subset $A = \{a, c\}$ is *(gr)-closed set but not $r$-closed set.

Theorem 2.4 For a topological space $(X, \tau)$, the following hold.

(i) Every *(gr)-closed set is g-closed.
(ii) Every *(gr)-closed set is rg-closed.
(iii) Every *(gr)-closed set is gs-closed.
(iv) Every *(gr)-closed set is gp-closed.
(v) Every *(gr)-closed set is gsp-closed.
(vi) Every *(gr)-closed set is gpr-closed.
(vii) Every *(gr)-closed set is gsp-closed.

Proof:

(i) Let $A$ be *(gr)-closed set in $(X, \tau).$ Let $U$ be an open set such that $A \subseteq U$. Since every open set is $\check{g}$-open set, $cl(A) \subseteq rcl(A) \subseteq U$, Therefore $cl(A) \subseteq U$. Hence $A$ is a g-closed set in $X$.

(ii) Let $A$ be *(gr)-closed set in $(X, \tau).$ Let $U$ be an r-open set such that $A \subseteq U$. Since every r-open set is $\bar{g}$-open set, $cl(A) \subseteq rcl(A) \subseteq U$, Therefore $cl(A) \subseteq U$. Hence $A$ is a rg-closed set in $X$.

(iii) Let $A$ be *(gr)-closed set in $(X, \tau).$ Let $U$ be an open set such that $A \subseteq U$. Since every open set is $\check{g}$-open set, $cl(A) \subseteq rcl(A) \subseteq U$, Therefore $cl(A) \subseteq U$. Hence $A$ is a gs-closed set in $X$.

(iv) Let $A$ be *(gr)-closed set in $(X, \tau).$ Let $U$ be an open set such that $A \subseteq U$. Since every open set is $\check{g}$-open set, $pcl(A) \subseteq rcl(A) \subseteq U$, Therefore $pcl(A) \subseteq U$. Hence $A$ is a gp-closed set in $X$.

(v) Let $A$ be *(gr)-closed set in $(X, \tau).$ Let $U$ be an open set such that $A \subseteq U$. Since every open set is $\check{g}$-open set, $spcl(A) \subseteq rcl(A) \subseteq U$, Therefore $spcl(A) \subseteq U$. Hence $A$ is a gsp-closed set in $X$.

(vi) Let $A$ be *(gr)-closed set in $(X, \tau).$ Let $U$ be an open set such that $A \subseteq U$. Since every open set is $\check{g}$-open set, $rcl(A) \subseteq rcl(A) \subseteq U$, Therefore $rcl(A) \subseteq U$. Hence $A$ is a gpr-closed set in $X$.

(vii) Let $A$ be *(gr)-closed set in $(X, \tau).$ Let $U$ be an open set such that $A \subseteq U$. Since every open set is $\check{g}$-open set, $cl(A) \subseteq rcl(A) \subseteq U$, Therefore $pcl(A) \subseteq U$. Hence $A$ is a gsp-closed set in $X$.

The converse of the above theorem need not be true as seen from the following examples.

Example 2.5

(i) Let $X = \{a, b, c, d\}$ with $\tau = \{\emptyset, X, \{a, b, c\}, \{d\}\}$. The g-closed sets are $\{\{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X, \emptyset\}$ and *(gr)-closed sets are $\{\{a, b, c\}, \{d\}\}$. The set $A = \{a, d\}$ is a g-closed set but not *(gr)-closed set in $X$.

(ii) Let $X = \{a, b, c, d\}$ with $\tau = \{\emptyset, X, \{a, b, c\}, \{a, b\}, \{a, c\}\}$. The rg-closed sets are $\{\{d\}, \{a, c\}, \{b, c\}, \{a, d\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, \emptyset\}$ and *(gr)-closed sets are $\{\{d\}, \{a, d\}, \{b, d\}, \{c, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$. The set $A = \{a, c\}$ is a rg-closed set but not *(gr)-closed set in $X$.

(iii) Let $X = \{a, b, c, d\}$ with $\tau = \{\emptyset, X, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}\}$. The gsp-closed sets are $\{\{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$. The set $A = \{a\}$ is a gsp-closed set but not *(gr)-closed set.

(iv) Let $X = \{a, b, c, d\}$ with $\tau = \{\emptyset, X, \{a, b, c\}, \{a, b\}, \{a, c\}\}$. The gp-closed sets are $\{\{c\}, \{d\}, \{a, d\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, \emptyset\}$ and *(gr)-closed sets are $\{\{a\}, \{a, b\}, \{a, c\}\}$. The set $A = \{b\}$ is a gp-closed set but not *(gr)-closed set in $X$.

(v) Let $X = \{a, b, c, d\}$ with $\tau = \{\emptyset, X, \{a, c, d\}, \{a\}\}$. The gpr-closed sets are $\{\emptyset, X, \{a\}\}$. The gsp-closed sets are $\{\{b\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, \emptyset\}$ and *(gr)-closed sets are $\{\{b\}$,
Remark 2.6
1. Closed set and gr-closed sets are independent of each other.
2. *(gr)-closed set and sg-closed sets are independent of each other.
3. *(gr)-closed set and gr-closed sets are independent of each other. It is shown by the following example.

Example 2.7
1. Let $X=\{a, b, c, d\}$ with topology $\tau=\{\emptyset, X, \{a\}, \{d\}, \{a, d\}, \{a, c, d\}, \{a, b, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{c\}, \{b, c\}, \{b, c, d\}, \{a, b\}, \{a, c\}, \{b\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, \{a, b, c, d\}, X, \emptyset\}$. The subset $\{a, c, d\}$ is *(gr)-closed set but not *sg-closed set in $X$.

Theorem 2.8
The finite union of the *(gr)-closed sets is a *(gr)-closed set.

Proof: Let A and B be *(gr)-closed set in $X$. Let $U$ be a $\tilde{g}$-open in $X$ such that $A \cup B \subseteq U$. Then $A \subseteq U$ and $B \subseteq U$. Since $A$ and $B$ are *(gr)-closed sets, $rcl(A) \subseteq U$ and $rcl(B) \subseteq U$. Hence $rcl(A \cup B) = rcl(A) \cup rcl(B) \subseteq U$. Therefore $A \cup B$ is a *(gr)-closed set, whenever $A$ and $B$ be *(gr)-closed set.

Remark 2.9
The finite intersection of two *(gr)-closed sets need not be a *(gr)-closed set.

Example 2.10
Let $X=\{a, b, c\}$ with $\tau=\{\emptyset, \{b\}, X\}$. Let $A=\{a, b\}$ be a *(gr)-closed set and $B=\{b, c\}$ be also a *(gr)-closed set. But $A \cap B=\{b\}$ is not *(gr)-closed set in $(X, \tau)$.

Theorem 2.11
If $A \subseteq B \subseteq rcl(A)$ and A is *(gr)-closed subset of $(X, \tau)$ then B is also a *(gr)-closed subset of $(X, \tau)$.

Proof: Let $U$ be a $\tilde{g}$-open subset such that $A \subseteq B \subseteq U$, since $A$ is *(gr)-closed subset of $(X, \tau)$, $rcl(A) \subseteq U$. by hypothesis $A \subseteq B \subseteq rcl(A)$, $rcl(A) = rcl(B)$. Hence $rcl(B) \subseteq U$ whenever $B \subseteq U$. Therefore $B$ is also a *(gr)-closed subset of $(X, \tau)$.

Theorem 2.12
Let A be a $\tilde{g}$-open subset of $(X, \tau)$. Then A is r-closed set if A is *(gr)-closed set.

Proof: Since A is $\tilde{g}$-open and *(gr)-closed in $(X, \tau)$, $rcl(A) \subseteq A$. Therefore A is r-closed set.

Theorem 2.13
Let $A \subseteq B \subseteq X$, where $B$ is $\tilde{g}$ - open and A is *(gr)-closed in B then A is a *(gr)-closed in X.

Proof: Let $U$ be a $\tilde{g}$-open subset of $X$ such that $A \subseteq U$. Since $A \subseteq U \cap B$, where $U \cap B$ is $\tilde{g}$-open is B and A is *(gr)-closed in B, $rcl(A) \subseteq U \cap B$ holds we have $rcl(A) \cap B \subseteq U \cap B$. Since $A \subseteq B$ we have $rcl(A) \subseteq rcl(B)$. Since $B$ is $\tilde{g}$-open and *(gr)-closed is X, by the above theorem, B is r-closed. Therefore $rcl(B) = B$. Thus $rcl(B) \subseteq B$ implies $rcl(A) = rcl(A) \cap B \subseteq U \cap B \subseteq U$. Hence A is *(gr)-closed is X.

Theorem 2.14
A subset A of X is a *(gr)-closed set in X if and only if $rcl(A) - A$ contains no non-empty $\tilde{g}$-closed set in X.

Proof: Suppose that F is a non-empty $\tilde{g}$-closed subset of $rcl(A) - A$. Now $F \subseteq rcl(A) - A$. Then $F \subseteq rcl(A) \cap A^C$. Therefore $F \subseteq rcl(A) \cap F \subseteq A^C$. Since $F$ is $\tilde{g}$-closed such that $A \subseteq F^C$ and A is *(gr)-closed, $rcl(A) \subseteq F^C$, ie, $F \subseteq rcl(A)^C$. Hence $F \subseteq rcl(A) \cap [rcl(A)]^C = \emptyset$. ie $F = \emptyset$. Thus $rcl(A) - A$ contains no non-empty $\tilde{g}$-closed set.

Conversely assume that $rcl(A) - A$ contains no non-empty $\tilde{g}$-closed set. Let $A \subseteq U$ and $U$ is $\tilde{g}$-open. Suppose that $rcl(A)$ is not contained in U. Then $rcl(A) \cap U^C$ is a non-empty $\tilde{g}$-closed set and contained in $rcl(A) - A$ which is a contradiction. Therefore $rcl(A) \subseteq U$ and hence A is *(gr)-closed set.

Theorem 2.15
For each $x \in X$, either $\{x\}$ is $\tilde{g}$-closed or $\{x\}^C$ is *(gr)-closed set in X.

Proof: If $\{x\}$ is not $\tilde{g}$-closed, then only $\tilde{g}$-open set containing $\{x\}$ is X. Thus $rcl(\{x\})^C$ is contained in X and hence $\{x\}^C$ is *(gr)-closed set in X.

Remark 2.16

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For the subsets defined above, we have the following implications.

4. *(gr)-OPEN SETS

**Definition 4.1** A subset A of a topological space (X, τ) is called a star generalized regular open set (briefly *(gr)-open) if its complement is *(gr)-closed set.

**Theorem 4.2** A subset A of a topological space (X, τ) is *(gr)-open if and only if B ⊆ rint(A) where B is a ġ-closed subset of X and B ⊆ A.

**Proof:** **Necessity:** Let A be *(gr)-open in X with N ⊆ A, where N is ġ-closed. We have A ⊆ ġ-closed with A ⊆ N where N is ġ-open. Then rcl(A) ⊆ N implies N ⊆ X-rcl(A) = r int(X - A) = r int(A).

**Sufficiency:** Suppose B ⊆ r int(A) where B is ġ-closed in (X, τ) and B ⊆ A. Let A ⊆ M, where M is ġ-open. Hence M is M C ⊆ A, where M C is ġ-closed. Hence by assumption M C ⊆ r int(A), which implies (r int(A)) C ⊆ M. Therefore rcl(A) ⊆ M. Thus A is *(gr)-closed, implies A is *(gr)-open.

**Theorem 4.3** Every r-open set is *(gr)-open set.

**Proof:** Let A be an r-open set. Then X-A is r-closed set. Then by theorem 3.1.2, X-A is *(gr)-closed. Hence A is *(gr)-open set.

**Theorem 4.4** If r int(A) ⊆ B ⊆ A and A is a *(gr)-open subset of (X, τ) then B is also a *(gr)-open subset of (X, τ).

**Proof:** Let r int(A) ⊆ B ⊆ A implies A ⊆ B ⊆ rcl(A). It is known that A is *(gr)-closed. Hence by theorem 3.1.13. B C is *(gr)-closed. Therefore B is *(gr)-open.

**Theorem 4.5** If a subset A of a topological space (X, τ) is *(gr)-open in X then F = X, whenever F is ġ-open and r int(A) ⊆ A C ⊆ F.

**Proof:** Let A be *(gr)-open in X and F be ġ-open, r int(A) ∪ A C ⊆ F. This gives F ⊆ (X-r int(A)) ∩ A = rcl(A C) ∩ A = rcl(A C) - A C. Since F is ġ-closed and A C is *(gr)-open by theorem 2.14. we have F C = Ø. Thus F = X.
References


