FIXED POINT THEOREM FOR WEAKLY INWARD NONSELF ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN BANACH SPACES

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Abstract


For a smooth banach space E, let us assume that K is a nonempty closed convex subset of with P as a sunny nonexpansive retraction. Let T_1, T_2, T_3 : K → E be three weakly inward nonself asymptotically nonexpansive mappings with respect to P with three sequences

\[ k_n^{(1)} \in [1, \infty) \] satisfying \( \sum_{n=1}^{\infty} (k_n^{(i)} - 1) < \infty \), (i=1,2,3) and F(T_1) \cap F(T_2) \cap F(T_3) = \{x \in K : T_1x = T_2x = T_3x = x\}

respectively.

For any given x_1 \in K, suppose that \{x_n\} is a sequence generated iteratively by

\[ x_{n+1} = a_n x_n + b_n (PT_1)^n y_n + c_n (PT_2)^n y_n + d_n (PT_3)^n y_n \]

\[ y_n = a_n x_n + b_n (PT_1)^n y_n + c_n (PT_2)^n y_n + d_n (PT_3)^n y_n \]

where \( \{a_n\}, \{b_n\}, \{c_n\}, \{d_n\} \) for i=1,2,3 are sequences in \[ a, 1 - a \] for some a \in (0,1), satisfying \( a_n + b_n + c_n + d_n = 1 \) (i=1,2,3). some a \in (0,1), Under some suitable conditions, the strong and weak convergence theorems of \{x_n\} to a common fixed point of T_1, T_2 and T_3 are obtained.

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Nonself asymptotically nonexpansive mapping, Strong and weak convergence, Common fixed point.

1 INTRODUCTION

For a self-mapping T : K → K, nonexpansive mapping is defined as \[ || T x - T y || \leq || x - y || \] for all x, y \in K and asymptotically nonexpansive if there exists a sequence \{k_n\} \subset [1, \infty) with \( k_n \to 1 \) as \( n \to \infty \) such that for all n \in N , where N stands for set of natural number,
As a generalization of the class of nonexpansive maps, the class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [5] in 1972, who proved that if \( K \) is a nonempty bounded closed convex subset of a real uniformly convex Banach space and \( T \) is an asymptotically nonexpansive self-mapping of \( K \), then \( T \) has a fixed point. Recently, Chidume et al. [1] further generalized the concept of nonself asymptotically nonexpansive mapping defined as follows:

Definition 1.1. [2] Let \( K \) be a nonempty subset of real normed linear space \( E \). Let \( P : E \rightarrow K \) be the nonexpansive retraction of \( E \) onto \( K \). (1) A nonself mapping \( T : K \rightarrow E \) is called asymptotically nonexpansive if there exist sequences \( \{k_n\}_{n \geq 1} \) with \( k_n \rightarrow 1 \) as \( n \rightarrow \infty \) such that

\[
||T^n x - T^n y|| \leq k_n ||x - y|| \quad \text{for all } x, y \in K. \tag{1.1}
\]

(2) A nonself mapping \( T : K \rightarrow E \) is said to be uniformly \( L \)-Lipschitzian if there exists a constant \( L \geq 0 \) such that

\[
||T^n x - T^n y|| \leq L ||x - y|| \quad \text{for all } x, y \in K. \tag{1.2}
\]

By using the following iterative algorithm:

\[
x_{n+1} = P\left(\left(1-a_n\right)x_n + a_n T(P)^{n-1} x_n\right), \tag{1.5}
\]

\( \forall \ n \geq 1 \), some authors [2, 6, 7, 11] have studied the strong and weak convergence theorem for such mappings.

As a matter of fact, if \( T \) is a self-mapping, then \( P \) is a identity mapping. Thus (1.3) and (1.4) reduce to (1.1) and (1.2) as \( T \) is a self-mapping, respectively. In addition, if \( T : K \rightarrow E \) is asymptotically nonexpansive in light of (1.3) and \( P : E \rightarrow K \) is a nonexpansive retraction, then \( PT : K \rightarrow K \) is asymptotically nonexpansive in light of (1.1). Indeed, for all \( x, y \in K \) and \( n \geq 1 \), by (1.3), it follows that

\[
|| (PT)^n x - (PT)^n y|| = ||PT(PT)^{n-1} x PT(PT)^{n-1} y|| \leq || PT(PT)^{n-1} x - PT(PT)^{n-1} y|| \leq k_n ||x - y||. \tag{1.6}
\]

Conversely, it may not be true. Therefore, Zhou et al. [13] introduced the following generalized definition recently.

Definition 1.2. [9] Let \( K \) be a nonempty subset of real normed linear space \( E \). Let \( P : E \rightarrow K \) be a nonexpansive retraction of \( E \) onto \( K \).

(1) A nonself mapping \( T : K \rightarrow E \) is called asymptotically nonexpansive with respect to \( P \) if there exist sequences \( \{k_n\}_{n \geq 1} \) with \( k_n \rightarrow 1 \) as \( n \rightarrow \infty \) such that

\[
|| (PT)^n x - (PT)^n y|| \leq k_n ||x - y|| \quad \forall x, y \in K, \ n \geq 1. \tag{1.7}
\]

(2) A nonself mapping \( T : K \rightarrow E \) is said to be uniformly \( L \)-Lipschitzian with respect to \( P \) if there exists a constant \( L \geq 0 \) such that

\[
|| (PT)^n x - (PT)^n y|| \leq L ||x - y|| \quad \forall x, y \in K, \ n \geq 1. \tag{1.8}
\]

Furthermore, by studying the following iterative process:

\[
x_{n+1} = \alpha_n \beta_n + \beta_n (PT_1)^{n} x_n + \gamma_n (PT_2)^{n} x_n \quad \forall x_1 \in K, \ n \geq 1. \tag{1.9}
\]

where \( \{\alpha_n\}, \{\beta_n\} \), and \( \{\gamma_n\} \) are three sequences in \( [0, 1-\alpha] \) for some \( \alpha \in (0, 1) \), satisfying \( \alpha_n + \beta_n + \gamma_n = 1 \). Zhou et al. [3] obtained some strong and weak convergence theorems for common fixed points of nonself asymptotically nonexpansive mappings with respect to \( P \) in uniformly convex Banach spaces. As a consequence, the main results of Chidume et al. [1] are deduced.

Inspired and motivated by those work mentioned above and three step iteration method proposed by Noor [8], in this paper, we construct a three step iteration scheme for approximating common fixed points of three nonself asymptotically nonexpansive mappings with respect to \( P \) and to prove some strong and weak convergence theorems for such mappings in uniformly convex Banach spaces.
2. PRELIMINARIES

Let $K$ be a nonempty closed convex subset of a real uniformly convex Banach space $E$ with retraction $P$. Let $T_1, T_2, T_3 : K \to E$ be three nonself asymptotically nonexpansive mappings with respect to $P$. For approximating common fixed points of such mappings, we further generalize the iteration scheme (1.9) as follows:

$$x_{n+1} = a_{n1}x_n + b_{n1}(PT_1)^n y_n + c_{n1} (PT_2)^n y_n + d_{n1} (PT_3)^n y_n$$

$$y_n = a_{n2}x_n + b_{n2}(PT_1)^n y_n + c_{n2} (PT_2)^n y_n + d_{n2} (PT_3)^n y_n$$

$$z_n = a_{n3}x_n + b_{n3}(PT_1)^n y_n + c_{n3} (PT_2)^n y_n + d_{n3} (PT_3)^n y_n$$

Where $\{a_{ni}\}, \{b_{ni}\}, \{c_{ni}\}, \{d_{ni}\}, i = \{1,2,3\}$ are sequences in $[0,1]$ satisfying

$$a_{ni} + b_{ni} + c_{ni} + d_{ni} \text{ for } \{1,2,3\}$$

Let $E$ be a Banach space with dimension $E \geq 2$. The modulus of $E$ is the function $\delta_E(\varepsilon) : (0,2) \to [0,1]$ defined by

$$\delta_E(\varepsilon) = \inf \left\{ 1 - \| \frac{1}{2}(x+y) \| ; \| x \|, \| y \| = 1, \varepsilon = \| x - y \| \right\}$$

A Banach space $E$ is uniformly convex if and only if $\delta_E(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$. Let $E$ be a Banach space and $S(E) = \{ x \in E : x = 1 \}$. The space $E$ is said to be smooth if

$$\lim_{t \to 0} \frac{\| x + ty \| - \| x \|}{t}$$

exists for all $x, y \in S(E)$.

A subset $K$ of $E$ is said to be retract if there exists continuous mapping $P : E \to K$ such that $Px = x$ for all $x \in K$. A mapping $P : E \to E$ is said to be a retraction if $P^2 = P$. Let $C$ and $K$ be subsets of a Banach space $E$. A mapping $P$ from $C$ into $K$ is called sunny if $P(Px + t(x-Px)) = Px$ for $x \in C$ with $Px = z$ for every $z \in R(P)$ (the range of $P$). It is well-known that every closed convex subset of a uniformly convex Banach space is a retract. For any $x \in K$, the inward set $I_k(x)$ is defined as follows: $I_k(x) = \{ y \in E : y = x + \lambda (z-x), z \in K, \lambda \geq 0 \}$. A mapping $T : K \to E$ is said to satisfy the inward condition if $T_x \in I_k(x)$ for all $x \in K$. T is said to satisfy the weakly inward condition if, for each $x \in K, T_x \in \text{cl} I_k(x)$ (the closure of $I_k(x)$).

A Banach space $E$ is said to satisfy Opial’s condition if, for any sequence $\{x_n\}$ in $E$, $x_n \to x$ implies that

$$\limsup_{n \to \infty} |x_n - x| < \limsup_{n \to \infty} |x - y|$$

for all $y \in E$ with $y \neq x$, where, $x_n \to x$ denotes that $\{x_n\}$ converges weakly to $x$.

Let $K$ be a nonempty closed subset of a real Banach space $E$. $T : K \to E$ is said to be demicompact if, for any sequence $\{x_n\} \subset K$ with $\|x_n - Tx_n\| \to 0$ (as $n \to \infty$) there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges strongly to $x^* \in K$.

A mapping $T$ with domain $D(T)$ and range $R(T)$ in $E$ is said to be semi-closed at $p$ if whenever $\{x_n\}$ is a sequence in $D(T)$ such that $\{x_n\}$ converges weakly to $x^* \in D(T)$ and $\{Tx_n\}$ converges strongly to $p$, then $Tx^* = p$.

Lemma 2.1. [12] Let $\{a_n\}, \{\delta_n\}, \{b_n\}$ be sequences of nonnegative real numbers satisfying

$$a_n + 1 \leq (1 + \delta_n) a_n + b_n, \forall n \geq 1,$$  

$$\sum_{n=1}^{\infty} \delta_n < \infty \text{ and } \sum_{n=1}^{\infty} b_n < \infty \text{ then } \lim a_n \text{ exists.}$$

Lemma 2.2. [6] Let $E$ be a real uniformly convex Banach space and let $B_r(0)$ be the closed ball of $E$ with center at the origin and radius $r > 0$. Then, there exists a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that

$$\| \lambda x + \mu y + \gamma z \| \leq \lambda \| x \|^2 + \mu \| y \|^2 + \gamma \| z \|^2 - \lambda \mu g(\| x - y \|)$$

For all $x,y,z \in B_r(0)$ and $\lambda, \mu, \gamma \in [0,1]$ with $\lambda + \mu + \gamma = 1$.

Lemma 2.3. [7] Let $E$ be a real smooth Banach space, let $K$ be a nonempty closed convex subset of $E$ with $P$ as a sunny nonexpansive retraction, and let $T : K \to E$ be a mapping satisfying weakly inward condition. Then $F(PT) = F(T)$.

Lemma 2.4. [3] Let $E$ be a uniformly convex Banach space and $K$ a nonempty closed convex subset of $E$. Let $T : K \to K$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\}$.
Lemma 3.1. Let $K$ be a nonempty closed convex subset of a real uniformly convex Banach space $E$. Let, $T_1, T_2, T_3 : K \to E$ be three nonself asymptotically non expansive mappings with respect to $P$ with three sequences $\{k_{n}^{(1)}\}, \{k_{n}^{(2)}\}, \{k_{n}^{(3)}\}, c \in [1, \infty)$ satisfying $\sum_{n=1}^{\infty} (k_{n}^{(i)} - 1) < \infty$ $(i = 1, 2, 3)$, respectively. Suppose that $\{x_n\}$ is defined by (2.1), where $\{a_n\}, \{b_n\}, \{c_n\}$ and $\{d_n\}, (i = 1, 2, 3)$ are sequences in $[m, 1 - m]$ for some $m \in (0, 1)$. If $F = F(T_1) \cap F(T_2) \cap F(T_3) = \emptyset$, then

\[
\begin{align*}
(1) & \quad \lim_{n \to \infty} ||x_n - q|| \text{ exists}, \forall q \in F; \\
(2) & \quad \lim_{n \to \infty} d(x_n, F) \text{ exists, where } d(x_n, F) = \inf_{q \in F} ||x_n - q||; \\
(3) & \quad \lim_{n \to \infty} ||x_n - (PT_i)x_n|| = 0 \quad (i = 1, 2, 3)
\end{align*}
\]

Proof: Setting $k_n = \max\{k_{n}^{(1)}, k_{n}^{(2)}, k_{n}^{(3)}\}$ since, $\sum_{n=1}^{\infty} (k_{n}^{(i)} - 1) < \infty$ $(i = 1, 2, 3)$

So, $\sum_{n=1}^{\infty} (k_{n}^{(i)} - 1) < \infty$.

For any $q \in F$, by (2.1) we have

\[
\begin{align*}
||x_n - q|| & \leq a_{n_1}||x_n - q|| + b_{n_1}((PT_1^n)x_n - q) + c_{n_1}((PT_2^n)x_n - q) + d_{n_1}((PT_3^n)x_n - q) \\
& \leq a_{n_3}||x_n - q|| + b_{n_3}k_n||x_n - q|| + c_{n_3}k_n||x_n - q|| + d_{n_3}k_n||x_n - q|| \\
& \leq k_n||x_n - q||. \\
\end{align*}
\]

By (2.1) and (3.1) we have

\[
\begin{align*}
||y_n - q|| & = ||a_{n_2}(x_n - q) + b_{n_2}((PT_1^n)x_n - q) + c_{n_2}((PT_2^n)x_n - q) + d_{n_2}((PT_3^n)x_n - q)|| \\
& \leq k_n^2||x_n - q||. \\
\end{align*}
\]

And hence, it follows from (2.1) and (3.2)

\[
\begin{align*}
||x_{n+1} - q|| & = ||a_{n_1}(x_n - q) + b_{n_1}((PT_1^n)x_n - q) + c_{n_1}((PT_2^n)y_n - q) + d_{n_1}((PT_3^n)y_n - q)|| \\
& \leq k_n^3||x_n - q||
\end{align*}
\]

Lemma 3.2. Assume, by conclusion of (1), $\lim_{n \to \infty} ||x_n - q|| = d$ and from lemma (2.2), we have,

\[
\begin{align*}
||x_n - q|| & \leq a_{n_1}||x_n - q|| + b_{n_1}((PT_1^n)y_n - q) + c_{n_1}((PT_2^n)y_n - q) + d_{n_1}((PT_3^n)y_n - q) \\
& \leq a_{n_1}||x_n - q|| + b_{n_1}||((PT_1^n)y_n - q)||^2 + c_{n_1}||((PT_2^n)y_n - q)||^2 + d_{n_1}||((PT_3^n)y_n - q)||^2 \\
& \leq a_{n_1}||x_n - q||^2 + b_{n_1} + c_{n_1} + d_{n_1}k_n^2||x_n - (PT_1^n)y_n|| \\
& \leq (a_{n_1} + b_{n_1} + c_{n_1} + d_{n_1}k_n^2)||x_n - q||^2 - m_3g_1||x_n - (PT_1^n)y_n|| \\
& \leq k_n^4||x_n - q||^2 - m_3g_1||x_n - (PT_1^n)y_n||
\end{align*}
\]

which implies that $g_1||x_n - (PT_1^n)y_n|| \to 0$ as $n \to \infty$. Since $g_1 : [0, \infty) \to [0, \infty)$ with $g_1(0) = 0$ is a continuous strictly increasing convex function, it follows that

\[
\lim_{n \to \infty} ||x_n - (PT_1^n)y_n|| = 0
\]

Similarly we have,

\[
\lim_{n \to \infty} ||x_n - (PT_2^n)y_n|| = 0
\]

And
\[
\lim_{n \to \infty} \|x_n - (PT_3^n)^n y_n\| = 0 \quad \text{(3.6)}
\]

Noting that,
\[
\|x_n - q\| = \|x_n - (PT_1^n) y_n\| + \| (PT_1^n) y_n - q\| \\
\leq \|x_n - (PT_1^n) y_n\| + k_n \|y_n - q\|
\]
we obtain from (3.4) that, by taking liminf on both sides in the inequality above,
\[
d = \liminf_{n \to \infty} \|x_n - q\| \leq \liminf_{n \to \infty} k_n \|y_n - q\| = \liminf_{n \to \infty} \|y_n - q\|
\]
In addition, it follows from (3.2) that \( \limsup_{n \to \infty} \|y_n - q\| \leq d, \) thus
\[
\lim_{n \to \infty} \|y_n - q\| = d \quad \text{(3.7)}
\]

Hence, by (2.1), (3.1), (3.6) and Lemma 2.2, we have
\[
\|y_n - q\|^2 = |a_{n2}(x_n - q) + b_{n2}((PT_1^n) y_n - q) + c_{n2}((PT_2^n) y_n - q) + d_{n2}((PT_3^n) y_n - q)|^2 \\
\leq a_{n2}^2 \|x_n - q\|^2 + b_{n2}^2 \|((PT_1^n) y_n - q)\|^2 + c_{n2}^2 \|((PT_2^n) y_n - q)\|^2 + d_{n2}^2 \|((PT_3^n) y_n - q)\|^2 \\
\leq a_{n2}^2 \|x_n - q\|^2 + b_{n2}^2 \|((PT_1^n) y_n - q)\|^2 + c_{n2}^2 \|((PT_2^n) y_n - q)\|^2 + d_{n2}^2 \|((PT_3^n) y_n - q)\|^2 \\
\leq a_{n2}^2 \|x_n - q\|^2 + (b_{n2} + c_{n2} + d_{n2}) k_n \|y_n - q\|^2 - m^2 g_2 \|x_n - (PT_1^n) y_n\|^2 \\
\leq (a_{n2} + b_{n2} + c_{n2} + d_{n2}) k_n \|x_n - q\|^2 - m^2 g_2 \|x_n - (PT_1^n) y_n\|^2 \\
\leq k_n \|x_n - q\|^2 - m^2 g_1 \|x_n - (PT_1^n) y_n\|^2
\]
which implies that \( g_2 \|x_n - (PT_1^n) y_n\| \to 0 \) as \( n \to \infty. \)

Since \( g_2 : [0, \infty) \to [0, \infty) \) with \( g_1(0) = 0 \) is a continuous strictly increasing convex function, it follows that
\[
\lim_{n \to \infty} \|x_n - (PT_1^n) y_n\| = 0 \quad \text{(3.8)}
\]

Similarly we have,
\[
\lim_{n \to \infty} \|x_n - (PT_2^n) y_n\| = 0 \quad \text{(3.9)}
\]

And
\[
\lim_{n \to \infty} \|x_n - (PT_3^n) y_n\| = 0 \quad \text{(3.10)}
\]
Furthermore, we claim that \(|x_{n+1} - x_n| \to 0\) as \(n \to \infty\). In fact, by (2.1), we have

\[
|x_{n+1} - x_n| = |b_{n1}((PT_1^n)x_n - x_n) + c_{n1}((PT_2^n)x_n - x_n) + d_{n1}((PT_3^n)x_n - x_n)|
\]

\[
\leq b_{n1}||((PT_1^n)x_n - x_n) + c_{n1}((PT_2^n)x_n - x_n) + d_{n1}((PT_3^n)x_n - x_n)||
\]

Hence, it follows from (3.4), (3.5) and (3.6)

\[
\lim_{n \to \infty} |x_{n+1} - x_n| = 0 \quad (3.15)
\]

Since any asymptotically nonexpansive mapping with respect to \(P\) must be uniformly \(L\)-Lipschitzian with respect to \(P\), where \(L = \sup_{n\geq1} \{k_n\} \geq 1\), we have,

\[
|x_{n+1} - (PT_i)x_{n+1}|
\]

\[
\leq |x_{n+1} - (PT_i)^n x_{n+1}| + |(PT_i)x_{n+1} - (PT_i)^n x_{n+1}|
\]

\[
\leq |x_{n+1} - (PT_i)^n x_{n+1}| + |(PT_i)x_{n+1} - (PT_i)^n x_{n+1}|
\]

\[
\leq |x_{n+1} - (PT_i)^n x_{n+1}| + |L||x_{n+1} - x_n|
\]

\[
\leq |x_{n+1} - (PT_i)^n x_{n+1}| + |L||x_{n+1} - x_n|
\]

Consequently, by (3.13), (3.14), and (3.15), it can be obtained that,

\[
\lim_{n \to \infty} |x_{n+1} - (PT_i)x_{n+1}| = 0 \quad (i=1,2,3) \quad (3.17)
\]

This completes the proof.

Theorem 3.2. Let \(K\) be a nonempty closed convex subset of a real uniformly convex and smooth Banach space \(E\) with \(P\) as a sunny nonexpansive retraction. Let \(T_1, T_2, T_3 : K \to E\) be three weakly inward nonself asymptotically nonexpansive mappings with respect to \(P\) with two sequences \(\{k_n^{(1)}\}, \{k_n^{(2)}\}, \{k_n^{(3)}\}\) \(c[1, \infty)\) satisfying \(\sum_{n=1}^{\infty} (k_n^{(i)} - 1) < \infty\), \(i = 1, 2, 3\) respectively. Suppose that sequence \(\{x_n\}\) defined by (2.1) where \(\{a_{ni}\}, \{b_{ni}\}, \{c_{ni}\}\) and \(\{d_{ni}\}, (i = 1, 2, 3)\) are sequences in \([m, 1 - m]\) for some \(m \in (0, 1)\).

If \(P(T_1)\) and \(P(T_2)\) and \(P(T_3)\) satisfy Condition (B) with respect to the sequence \(\{x_n\}\), i.e., there exists a nondecreasing function \(f: [0, \infty) \to [0, \infty)\) with \(f(0) = 0\) and \(f(r) > 0\) for all \(r \in (0, \infty)\) such that \(f(d(x_n, F))) \leq \max_{1 \leq i \leq 3} |x_n - (PT_i)x_n|\)

\[
f(d(x_n, F)) \leq \max_{1 \leq i \leq 3} |x_n - (PT_i)x_n|
\]

Taking \(\limsup\) as \(n \to \infty\) on both sides in the inequality above, we get

\[
\lim_{n \to \infty} f(d(x_n, F)) = 0
\]

which implies \(\lim_{n \to \infty} f(d(x_n, F)) = 0\), by the definition of the function \(f\).

Now we show that \(\{x_n\}\) is a Cauchy sequence. By (3.3), we may assume that \(\sum_{n=0}^{\infty} \delta_n = M \geq 0\), since \(\lim_{n \to \infty} d(x_n, F) = 0\), then for any \(\varepsilon > 0\), there exists a positive integer \(N\) such that \(d(x_n, F) < \frac{\varepsilon}{2e^{n/2}}\) for all \(n \geq N\). On the other hand, there exists a \(p \in F\) such that \(d(x_n, F) = d(x_n, F) < \frac{\varepsilon}{2e^{n/2}}\) because \(d(x_n, F) = \inf_{x \in F} d(x_n, F)\). Therefore, \(\lim_{n \to \infty} d(x_n, F) = 0\) and \(F\) is closed. Thus, for any \(n \in N\), it follows from (3.3) that

\[
\left| x_{n+1} - x_n \right| = (1+\delta_n) \left| x_{n+1} - x_n \right| \leq \prod_{i=1}^{N_n} \left| x_{n+1} - x_n \right| \leq e^{\sum_{i=1}^{N_n} (1+\delta_i)} \left| x_{n} - p \right|
\]

\[
\leq e^{\sum_{i=1}^{N_n} (1+\delta_i)} \left| x_{n} - p \right|
\]
\[ \leq d(M_n, p) \]

Hence, for any \( m, n > N \)

\[ ||x_n - x_m|| ^2 \leq ||x_n - p||^2 + ||x_m - p||^2 - 2 \epsilon e^{\alpha m} ||x_n - p|| < \epsilon \]

This implies that \( \{x_n\} \) is a Cauchy sequence. Thus, there exists a \( x \in K \) such that \( x_n \to x \) as \( n \to \infty \), since \( E \) is complete. Then, \( \lim_{n \to \infty} d(x_n, F) = 0 \) yields that \( d(x, F) = 0 \). Further, it follows from the closedness of \( F \) that \( x \in F \). This completes the proof.

**Theorem 3.3.** Let \( K \) be a nonempty closed convex subset of a uniformly convex and smooth Banach space \( E \) satisfying Opial's condition with \( P \) as a sunny nonexpansive retraction. Let \( T_1, T_2, T_3 : K \to E \) be two weakly inward nonself asymptotically nonexpansive mappings with respect to \( P \) with two sequences \( \{k_{n1}\}, \{k_{n2}\}, \{k_{n3}\} \subset [1, \infty) \) satisfying

\[ \sum_{n=1}^{\infty} (k_{ni} - 1) < \infty \quad (i=1, 2, 3) \]

respectively. Suppose that \( \{x_n\} \) defined by (2.1) where \( \{a_{ni}\}, \{b_{ni}\}, \{c_{ni}\} \) and \( \{d_{ni}\} \), \( i = 1, 2, 3 \) are sequences in \([m, 1-m)\) for some \( m \in (0, 1) \).

If \( F := F(T_1) \cap F(T_2) \cap F(T_3) = \emptyset \), then \( \{x_n\} \) converges weakly to some common fixed point of \( T_1, T_2 \) and \( T_3 \).

**Proof.** For any \( q \in F \), by Lemma 3.1, we know that \( \lim_{n \to \infty} ||x_n - q|| \) exists. We now prove that \( \{x_n\} \) has a unique weakly subsequential limit in \( F \). First of all, since \( P(T_1), P(T_2) \) and \( P(T_3) \) are self-mappings from \( K \) into itself, therefore, Lemmas 2.3, 2.4, and 3.1 guarantee that each weakly subsequential limit of \( \{x_n\} \) is a common fixed point of \( T_1, T_2 \) and \( T_3 \). Secondly, Opial's condition guarantees that the weakly subsequential limit of \( \{x_n\} \) is unique. Consequently, \( \{x_n\} \) converges weakly to a common fixed point of \( T_1, T_2 \) and \( T_3 \). This completes the proof.

**REFERENCES**


