Soft g*-closed sets in Soft Topological Spaces

A.Devika, L.Elvina Mary

1Department of Mathematics, PSG College of Arts and Science, Coimbatore, India.
2Department of Mathematics, Nirmala College for Women, Coimbatore, India.

Abstract

The focus of this paper is to introduce Soft g*-closed sets in soft topological spaces which are defined over an initial universe with a fixed set of parameters. Moreover we investigate the relationship of soft g*-closed sets with other soft closed sets. Further the new separation axioms namely soft $T^*_1$-space and soft $T^*_2$-space are introduced and its basic properties are discussed.

Keywords: soft g*-closed, soft g*-open, soft $T^*_1$-space, soft $T^*_2$-space

1.INTRODUCTION

Molodstov[6] initiated the concept of soft set theory as a new mathematical tool for dealing with uncertainty problems. Soft systems provide a general framework with the involvement of parameters. In recent years the development in the field of soft set theory and its application has been taking place in a rapid pace. Muhammad Shabir and Munazza Naz [7] introduce the notion of soft topological spaces which are defined over an initial universe with a fixed set of parameters.

Levine [4] introduced g-closed sets in general topology. Kannan [3] introduced soft g-closed sets in soft topological spaces. In the present study, we introduce some new concepts in soft topological such as soft g*-closed sets and soft g*-open sets and derive some of its properties.

2.PRELIMINARIES

Definition:2.1[6] Let $U$ be the initial universe and $P(U)$ denote the power set of $U$. Let $E$ denote the set of all parameters. Let $A$ be a non-empty subset of $E$. A pair $(F,A)$ is called a soft set over $U$, where $F$ is a mapping given by $F:A \rightarrow P(U)$. In other words, a soft set over $U$ is a parameterized family of subsets of the universe $U$. For $\varepsilon \in A, F(\varepsilon)$ may be considered as the set $\varepsilon$—approximate elements of the soft set $(F,A)$. 
Definition :2.2[6] For two soft sets \((F,A)\) and \((G,B)\) over a common universe \(U\), we say that \((F,A)\) is a soft subset of \((G,B)\) if (1) \(A \subseteq B\) and (2) for all \(e \in A\), \(F(e)\) and \(G(e)\) are identical approximations. We write \((F,A) \subseteq (G,B)\). \((F,A)\) is said to be a soft super set of \((G,B)\), if \((G,B)\) is a soft subset of \((F,A)\). We denote it by \((F,A) \supseteq (G,B)\).

Definition :2.3[5] Two soft sets \((F,A)\) and \((G,B)\) over a common universe \(U\) are said to be soft equal if \((F,A)\) is soft subset of \((G,B)\) and \((G,B)\) is a soft subset of \((F,A)\).

Definition :2.4[5] The union of two soft sets of \((F,A)\) and \((G,B)\) over the common universe \(U\) is the soft set \((H,C)\) where \(C = A \cup B\) and for all \(e \in C\),
\[
H(e) = \begin{cases} 
F(e) & \text{if } e \in A - B \\
G(e) & \text{if } e \in B - A \\
F(e) \cup G(e) & \text{if } e \in A \cap B
\end{cases}
\]
We write \((F,A) \cup (G,B) = (H,C)\)

Definition :2.5[6] The intersection \((H,C)\) of two soft sets of \((F,A)\) and \((G,B)\) over the common universe \(U\) denoted \((F,A) \cap (G,B)\), is defined as \(C = A \cap B\) and \(H(e) = F(e) \cap G(e)\) for all \(e \in C\).

Definition :2.6[7] Let \(\tau\) be the collection of soft sets over \(X\), then \(\tau\) is said to be a soft topology on \(X\) if (1) \(\emptyset, X\) belong to \(\tau\), (2) the union of any number of soft sets in \(\tau\) belongs to \(\tau\), (3) the intersection of any two soft sets in \(\tau\) belongs to \(\tau\). The triplet \((X, \tau, E)\) is called a soft topological space over \(X\). Let \((X, \tau, E)\) be a soft space over \(X\), then the members of \(\tau\) are said to be soft open sets in \(X\).

Definition :2.7[8] A subset \((A,E)\) of a topological space \(X\) is called soft regular closed (soft r-closed) if \(\text{cl}(\text{int}(A,E)) = (U,E)\). The complement of soft regular closed is soft regular set.

Definition :2.8[9] The finite union of soft regular open sets is said to be soft \(\pi\)-open. The complement of soft \(\pi\)-open is said to be soft \(\pi\)-closed.

Definition :2.9[3] A subset \((A,E)\) of a topological space \(X\) is called a soft generalized closed (soft g-closed) if \(\text{cl}(A,E) \subseteq (U,E)\) whenever \((A,E) \subseteq (U,E)\) and \((U,E)\) is soft open in \(X\).

Definition :2.10[1] A subset \((A,E)\) of a topological space \(X\) is called a soft regular closed (soft r-closed) if \(\text{cl}(\text{int}(A,E)) = (A,E)\). The complement of soft regular closed set is soft regular open set.
Definition : 2.11[9] A subset \((A,E)\) of a topological space \(X\) is called a soft \(\pi g\) -closed in a soft topological space \((X, \tau, E)\) if \(\text{cl}(A,E) \subset (U,E)\) whenever \((A,E) \subset (U,E)\) and \((U,E)\) is soft \(\pi\) -open in \(X\).

Definition : 2.12[9] A subset \((A,E)\) of a topological space \(X\) is called a soft \(g\) \(\pi\) -closed in a soft topological space \((X, \tau, E)\) if \(\text{scl}(A,E) \subset (U,E)\) whenever \((A,E) \subset (U,E)\) and \((U,E)\) is soft \(\pi\) -open in \(X\).

Definition : 2.13[9] A subset \((A,E)\) of a topological space \(X\) is called a soft \(g\) \(\pi\) \(\alpha\) -closed in a soft topological space \((X, \tau, E)\) if \(\text{pcl}(A,E) \subset (U,E)\) whenever \((A,E) \subset (U,E)\) and \((U,E)\) is soft regular open in \(X\).

Definition : 2.14[8] A subset \((A,E)\) of a topological space \(X\) is called a soft regular generalised-closed(soft \(rg\)-closed) in a soft topological space \((X, \tau, E)\), if \(\text{cl}(A,E) \subset (U,E)\) whenever \((A,E) \subset (U,E)\) and \((U,E)\) is soft regular open in \(X\).

Definition : 2.15[1] A subset \((A,E)\) of a topological space \(X\) is called a soft generalised-semi closed(soft \(gs\)-closed) in a soft topological space \((X, \tau, E)\), if \(\text{scl}(A,E) \subset (U,E)\) whenever \((A,E) \subset (U,E)\) and \((U,E)\) is soft open in \(X\).

Definition : 2.16[1] A subset \((A,E)\) of a topological space \(X\) is called a soft \(\alpha\) generalised-closed(soft \(\alpha\) \(g\)-closed) in a soft topological space \((X, \tau, E)\), if \(\alpha \text{ cl}(A,E) \subset (U,E)\) whenever \((A,E) \subset (U,E)\) and \((U,E)\) is soft open in \(X\).

Definition : 2.17[1] A subset \((A,E)\) of a topological space \(X\) is called a soft generalized \(\beta\) closed(soft \(g\) \(\beta\)-closed) in a soft topological space \((X, \tau, E)\), if \(\beta \text{ cl}(A,E) \subset (U,E)\) whenever \((A,E) \subset (U,E)\) and \((U,E)\) is soft open in \(X\).

3.SOFT \(g^*\)-CLOSED SET

Definition : 3.1 A subset \((F,E)\) of a soft topological space \((X, \tau, E)\) is called a soft \(g^*\) -closed set, if \(\text{cl}(F,E) \subset (U,E)\) whenever \((F,E) \subset (U,E)\) and \((U,E)\) is \(g\)-open in \(X\).

Proposition : 3.2

i) Every soft closed set is soft \(g^*\)-closed.

ii) Every soft \(g^*\)-closed set is soft \(g\)-closed.

iii) Every soft \(g^*\)-closed set is soft \(gs\)-closed.
iv) Every soft $g^*$-closed set is soft $\pi g$-closed.

v) Every soft $g^*$-closed set is soft $g\beta$-closed.

vi) Every soft $g^*$-closed set is soft $rg$-closed.

vii) Every soft $g^*$-closed set is soft $\alpha g$-closed.

viii) Every soft $g^*$-closed set is soft $\pi g\beta$-closed.

ix) Every soft $g^*$-closed set is soft gpr-closed.

x) Every soft $g^*$-closed set is soft $gb$-closed.

Proof:

i) Let $(F,E)$ be soft closed set in $(X, \tau, E)$ and $(U,E)$ be soft g-open in $X$, such that $(F,E) \subset (U,E)$. Then $cl(F,E) = (F,E) \subset (U,E)$. Hence $(F,E)$ is soft $g^*$-closed.

ii) Let $(F,E)$ be soft $g^*$-closed set in $(X, \tau, E)$. Let $(U,E)$ be soft open in $X$ such that $(F,E) \subset (U,E)$. Since every soft open set is soft g-open, we have $cl(F,E) \subset (U,E)$. Hence $(F,E)$ is soft $g^*$-closed.

iii) Let $(F,E)$ be soft $g^*$-closed set in $(X, \tau, E)$. Let $(U,E)$ be soft open in $X$ such that $(F,E) \subset (U,E)$. Since every soft open set is soft g-open, we have $cl(F,E) \subset (U,E)$. But $scl(F,E) \subset cl(F,E) \subset (U,E)$. Hence $(F,E)$ is soft $gs$-closed.

iv) Proof is obvious and straightforward.

v) Proof is obvious and straightforward.

vi) Let $(F,E)$ be soft $g^*$-closed set in $(X, \tau, E)$. Let $(U,E)$ be soft regular open in $X$ such that $(F,E) \subset (U,E)$. Since every soft regular open set is soft g-open, we have $cl(F,E) \subset (U,E)$. Hence $(F,E)$ is soft $rg$-closed.

vii) Let $(F,E)$ be soft $g^*$-closed set in $(X, \tau, E)$. Let $(U,E)$ be soft open in $X$ such that $(F,E) \subset (U,E)$. Since every soft open set is soft $\alpha g$-open, we have $cl(F,E) \subset (U,E)$. But $\alpha cl(F,E) \subset cl(F,E) \subset (U,E)$. Hence $(F,E)$ is soft $\alpha g$-closed.

viii) Let $(F,E)$ be soft $g^*$-closed set in $(X, \tau, E)$. Let $(U,E)$ be soft $\pi$-open in $X$ such that $(F,E) \subset (U,E)$. Since every soft $\pi$-open set is soft g-open, we have $cl(F,E) \subset (U,E)$. But $scl(F,E) \subset cl(F,E) \subset (U,E)$. Hence $(F,E)$ is soft $\pi sg$-closed.
ix) Let \((F, E)\) be soft \(g^*\)-closed set in \((X, \tau, E)\). Let \((U, E)\) be soft regular open in \(X\) such that \((F, E) \subseteq (U, E)\). Since every soft regular open set is soft \(g\)-open, we have \(\text{cl}(F, E) \subseteq (U, E)\). But \(\text{pcl}(F, E) \subseteq \text{cl}(F, E) \subseteq (U, E)\). Hence \((F, E)\) is soft \(g_{pr}\)-closed.

x) Let \((F, E)\) be soft \(g^*\)-closed set in \((X, \tau, E)\). Let \((U, E)\) be soft \(\pi\)-open in \(X\) such that \((F, E) \subseteq (U, E)\). Since every soft \(\pi\)-open set is soft \(g\)-open, we have \(\text{cl}(F, E) \subseteq (U, E)\). But \(\text{sbcl}(F, E) \subseteq \text{cl}(F, E) \subseteq (U, E)\). Hence \((F, E)\) is soft \(\pi\) \(gb\)-closed.

**Theorem 3.3** Every finite union of soft \(g^*\)-closed set is soft \(g^*\)-closed

**Proof:** Let \((F, A)\) and \((G, B)\) be two soft \(g^*\)-closed subset of \(X\). Let \((U, E)\) be a soft \(g\)-open set in \((X, \tau, E)\), such that \((F, A) \cup (G, B) \subseteq (U, E)\). Then \(\text{cl}(F, A) \subseteq (U, E)\) and \(\text{cl}(G, B) \subseteq (U, E)\). Therefore \(\text{cl}((F, A) \cup (G, B)) \subseteq \text{cl}(F, A) \cap \text{cl}(G, B) \subseteq (U, E)\). This implies \(\text{cl}((F, A) \cup (G, B)) \subseteq (U, E)\). Hence \((F, A) \cup (G, B)\) is soft \(g^*\)-closed.

**Theorem 3.4** If \((F, E)\) is a soft \(g^*\)-closed set of \(X\) such that \((F, E) \subseteq (G, E) \subseteq \text{cl}(F, E)\), then \((G, E)\) is a soft \(g^*\)-closed.

**Proof:**

Let \((G, E) \subseteq (U, E)\) where \((U, E)\) is soft \(g\)-open. Then \((F, E) \subseteq (G, E)\) implies \((F, E) \subseteq (U, E)\). Since \((F, E)\) is a soft \(g^*\)-closed set, \(\text{cl}(F, E) \subseteq (U, E)\). Given \((G, E) \subseteq \text{cl}(U, E)\). Hence \(\text{cl}(G, E) \subseteq \text{cl}(\text{cl}(F, E)) \subseteq \text{cl}(F, E) \subseteq (U, E)\) which implies \(\text{cl}(G, E) \subseteq (U, E)\). Therefore \((G, E)\) is a soft \(g^*\)-closed set.
Theorem :3.5 A soft set \((G,E)\) is soft g*-closed if and only if \(\text{cl}(G,E) - (G,E)\) contains only null soft g-closed set.

**Proof:** **Necessity part** Let \((G,E)\) be a soft g*-closed set. Let \((F,E)\) be soft g-closed such that \((F,E) \subset \text{cl}(G,E) - (G,E)\). Then \((F,E) \subset \text{cl}(G,E)\) and \((F,E) \subset (G,E)^C\). This implies \((G,E) \subset (F,E)^C\). Then \(\text{cl}(G,E) \subset (F,E)^C\) as \((F,E)^C\) is a soft g-open set. This implies \((F,E) \subset (\text{cl}(G,E))^C\). Therefore \((F,E) \subset \text{cl}(G,E) \cap (\text{cl}(G,E))^C\). Hence \((F,E)\) is a null soft g-closed set.

**Theorem :3.6** If \((A,E)\) is soft g-open and soft g*-closed, then \((A,E)\) is soft closed.

**Proof:** Obvious

4.SOFT g*-OPEN SET

**Definition :4.1** A subset \((A,E)\) of a topological space \(X\) is called soft g*-open in a soft topological space \((X, \tau, E)\), if \((F,E) \subset \text{int}(A,E)\) whenever \((F,E) \subset (A,E)\) and \((F,E)\) is soft g-closed in \(X\).

**Theorem :4.2** If \((A,E)\) is a soft g*-open set of \(X\) and \(\text{int}(A,E) \subset (B,E)\), then \((B,E)\) is also soft g*-open set of \(X\).

**Proof:** Let \((A,E)\) be soft g*-open in \(X\). Suppose \((G,E)\) is soft g-closed set such that \((G,E) \subset (B,E)\). By assumption, \((B,E) \subset (A,E)\) we have \((G,E) \subset (A,E)\). Since \((A,E)\) is soft g*-open set, \((G,E) \subset \text{int}(A,E)\). Then \(\text{int}(\text{int}(A,E)) \subset (B,E)\) implies that \(\text{int}(A,E) \subset \text{int}(B,E)\). Hence \((G,E) \subset \text{int}(A,E) \subset \text{int}(B,E)\) implies that \((G,E) \subset \text{int}(B,E)\). Then \((B,E)\) is soft g*-open set of \(X\).

**Theorem :4.3** If \((F,A)\) and \((G,B)\) are soft g*-open sets, then \(((F,A) \cap (G,B))\) is also soft g*-open set

**Proof:** Let \((F,A)\) and \((G,B)\) be soft g*-open sets. Suppose \((H,E)\) is soft g-closed set such that \((H,E) \subset ((F,A) \cap (G,B))\). Then \((H,E) \subset (F,A)\) and \((H,E) \subset (G,B)\). Since \((F,A)\) and \((G,B)\) are soft g*-open sets, \((H,E) \subset \text{int}(F,A)\) and \((H,E) \subset \text{int}(G,B)\). Therefore \((H,E) \subset \text{int}(F,A) \cap \text{int}(G,B)\). Thus \((H,E) \subset \text{int}((F,A) \cap (G,B))\). Hence \(((F,A) \cap (G,B))\) is a soft g*-open set.
5.SOFT $T^*_1$-SPACE AND SOFT $T^*_2$-SPACE

**Definition:** 5.1 A Soft topological space $(X, \tau , E)$ is called a soft $T^*_1$ space if every $g^*$-closed set is soft closed.

**Theorem:** 5.2 Every soft $T^*_1$ space is soft $T^*_2$ space.

**Proof:** Let $(X, \tau , E)$ be a soft $T^*_1$ space and let $(A, E)$ be soft $g^*$-closed set in $(X, \tau , E)$. By proposition 3.2(ii), $(A, E)$ is soft $g^*$-closed. Since $(X, \tau , E)$ is soft $T^*_1$ space, $(A, E)$ is soft closed in $(X, \tau , E)$.

**Theorem:** 5.3 For a space $(X, \tau , E)$, the following conditions are equivalent

1) $(X, \tau , E)$ is a soft $T^*_1$ space

2) Every singleton of $X$ is either soft $g^*$-closed set or soft open.

**Proof:** (1) $\Rightarrow$ (2) Let $x \in X$ and suppose $\{x\}$ is not soft $g^*$-closed set of $(X, \tau , E)$. Then $X - \{x\}$ is not soft $g^*$-open. This implies $X$ is the only soft set containing $X - \{x\}$. So $X - \{x\}$ is a soft $g^*$-closed set of $(X, \tau , E)$. Since $(X, \tau , E)$ is a soft $T^*_1$ space, then $X - \{x\}$ is soft closed or equivalently $\{x\}$ is soft open in $(X, \tau , E)$.

(2) $\Rightarrow$ (1) Let $A$ be a soft $g^*$-closed set of $(X, \tau , E)$. Trivially $(A, E) \subseteq \text{cl}(A, E)$. Let $x \in \text{cl}(A, E)$. By (2), $\{x\}$ is either soft $g^*$-closed or soft open.

Case (i): Suppose $\{x\}$ is soft $g^*$-closed. If $x \notin A$, then $\text{cl}(A, E) - (A, E)$ contains a non-empty soft $g^*$-closed set. But this is not possible according to the theorem 3.5 as $A$ is a soft $g^*$-closed set. Therefore $x \in A$.

Case (ii): Suppose $\{x\}$ is soft open. Since $x \in \text{cl}(A, E)$, then $\{x\} \cap A \neq \emptyset$, so $x \in (A, E)$.

Therefore $x \in (A, E)$. So in any case $\text{cl}(A, E) \subseteq (A, E)$, thus $\text{cl}(A, E) = (A, E)$ or equivalently $(A, E)$ is a soft closed set of $(X, \tau , E)$.
Definition: 5.4 A Soft topological space \((X, \tau, E)\) is called a soft \(\overset{\ast}{T}\) space if every soft \(g^\ast\)-closed set is soft \(g\)-closed.

Theorem: 5.5 Every soft \(T_\overset{\ast}{2}\) space is soft \(\overset{\ast}{T}\) space.

Proof: Let \((X, \tau, E)\) be a soft \(T_\overset{\ast}{2}\) space. Let \((A, E)\) be soft \(g\)-closed set of \((X, \tau, E)\). Since \((X, \tau, E)\) is soft \(T_\overset{\ast}{2}\) space, \((A, E)\) is soft closed in \((X, \tau, E)\). By proposition 3.2(i), \((A, E)\) is soft \(g^\ast\)-closed. Therefore \((X, \tau, E)\) is a soft \(\overset{\ast}{T}\) space.

Theorem: 5.6 If \((X, \tau, E)\) is a soft \(\overset{\ast}{T}\) space, then \(x \in X\), \(
\{x\}\) is either soft closed or soft \(g^\ast\)-open.

Proof: Suppose \((X, \tau, E)\) is a soft \(\overset{\ast}{T}\) space. \(x \in X\) and assume that \(
\{x\}\) is not soft closed. Then \(X - \{x\}\) is not a soft open set. This implies \(X - \{x\}\) is a \(g\)-closed set since \(X\) is the only soft open set which contains \(X - \{x\}\). Since \((X, \tau, E)\), then \(X - \{x\}\) is a soft \(g^\ast\)-closed or equivalently \(
\{x\}\) is a soft \(g^\ast\)-open.

References

1. I.Arockiarani and A.Arockia Lancy, Generalized soft \(g\beta\) closed sets and soft \(gs\beta\) closed sets in soft topological spaces, International Journal of Mathematical Archive - 4(2), 2013, 17-23.